1 Old-fashioned (not MCMC) Monte Carlo

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For a good chapter on this topic look at

1. Ross, Sheldon M. (1989) **Introduction to Probability Models,** Fourth Edition, Chapter 11.

First, we should understand that is impossible to get something random from a machine. We used to call simulated variable pseudo-random numbers, but now just call them random numbers.

In this talk we will assume a method to compute independent random variables which are $U(0,1)$. We will use these variables to find variables with other distributions

I know two general methods for simulating other distributions

1. (Inverse transformation method) Let $U \sim U(0,1)$. Let $F(x)$ be a distribution function with inverse function $G(u)$ and let

$$X = G(U).$$

Then X has distribution function $F(x)$

Proof.

$$P(X \leq x) = P(G(U) \leq x) = P(U \leq F(x)) = F(x)$$

2. (Rejection algorithm) Let X be a random variable with density function $g(x)$ and let $f(x)$ be another density function such

$$\frac{f(x)}{g(x)} \leq c \text{ forall } x$$

Simulate a random variable from X from $g(x)$, and a $U \sim U(0,1)$. Accept this variable X and call it Y if $U \leq p(X) = f(X)/cg(X)$. Otherwise reject the X and start over. Then Y has density $f(x)$. For this algorithm the number of trials necessary to get a good is a geometric random variable with parameter $\frac{1}{c}$, so the expected number of iterations is c.

3. **Example (simulating an exponential random variable)**

   (a) For an exponential random variable with mean μ the distribution function is

   $$F(x) = 1 - \exp\left(\frac{x}{\mu}\right)$$

   Using the inverse function method we see that this can be simulated by letting

   $$X = -\mu \ln(1 - U)$$

   where $U \sim U(0,1)$. Note that $1 - U$ and U have the same distribution so that

   $$-\mu \ln(U)$$

   also has an exponential distribution

(b) (von-Neuman) Let U_i be independent $U_i \sim U(0,1)$. Let

$$N = \min\{n : U_{n-1} < U_n\}$$

If N is even,start over calling this a failed run. If N is odd, then let X =number of failed runs $+U_1$ in the first successful run. Then X is exponential with mean 1. To get an experiential with mean μ, let $X^* = \mu X$. Requires on average $e/(1-e) = 4.3$ random numbers

4. **Example** (simulating a standard normal random variable)

(a) We can invert the distribution function. It has no formula but is tabled.

(b) We can let

$$Z = \left(\sum_{i=1}^{1200} U_i - 600 \right) /10$$

By the central limit theorem this will have approximately a standard normal distribution

(c) (Box-Mueller)Let U_1 and U_2 be independent, $U_i \sim U(0,1)$. Let

$$X = (-2\ln(U_1))^{1/2} \cos(2\pi U_2)$$

$$Y = (-2\ln(U_1))^{1/2} \sin(2\pi U_2)$$

then X and Y are independent standard normals

(d) (polar) Let U_1 and U_2 be independent,$U_i \sim U(0,1)$. Let

$$V_i = 2U_i - 1,\ S = V_1^2 + V_2^2$$

If $S > 1$, then start over. Otherwise, let

$$X = \sqrt{\frac{2\ln S}{S}} V_1,\ Y = \sqrt{\frac{2\ln S}{S}} V_2$$

Then X and Y are independent standard normals. Expected number of iterations is $4/\pi = 1.273$

(e) Use the rejection algorithm to simulate the absolute value of a standard normal from an exponential with mean 1. Note that

$$\frac{f(x)}{g(x)} = \frac{\frac{2}{\sqrt{2\pi}} \exp\left(-x^2\right)}{\exp\left(-x\right)} \leq \sqrt{\frac{2e}{\pi}} = c \approx 1.32$$

Therefore, we simulate an exponential random variable X with mean 1. We keep it with probability $p(X) = f(X)/cg(X)$. After we have the absolute value of Z, we choose the sign at random. The expected number of iterations is c=1.32

5. **Example** (simulating a Poisson process with rate λ)

   Take a sequence of $X_i \sim \exp(\lambda^{-1})$. Then use the X_i as the interarrival times in the Poisson process. This generates a Poisson process with rate λ.

6. **Example** (simulating a Poisson random variable)

   (a) Use the inverse of the distribution function method

   (b) Generate a Poisson process as in the pervious example and let N be the number of arrivals by time 1. Then $X \sim Poi(\lambda)$. Equivalently, simulate $X_i \sim \exp(1/\lambda)$ Let

   $$N = \max\left\{n : \sum_{i=1}^{n} X_i \leq 1\right\}$$

7. **Example** (simulating a non-homogeneous Poisson process with rate $\lambda(t) \le \lambda$)

   Simulate the arrivals from a homogeneous Poisson process with rate λ. Accept an arrival at time t for the non-homogeneous with probability $\lambda(t)/\lambda$.

8. **Example** (simulating a two dimensional Poisson process with rate λ) For a circle of radius r about the origin, we want to simulate the number of points and their location. It can be shown that the number of points in the circle can be generated essential as in the one dimensional problem given above. That is, we let $X_1, X_2, ...$ be experiential with mean $1/\lambda$ and

$$N = \max\left\{n : \sum_{i=1}^{n} X_i \le \lambda\pi r^2\right\}$$

   This is the simulated number of points in the region.

It is known that the angles of the points are independently uniformly distributed over $(0, 2\pi)$, so we can choose the points to be put at

$$R_i \cos(2\pi U_i^*), R_i \sin(2\pi U_i^*)$$

where

$$R_1^2 = \frac{X_1}{\pi\lambda},\ R_2^2 = \frac{X_1 + X_2}{\pi\lambda}$$

$$R_3^2 = \frac{X_1 + X_2 + X_3}{\pi\lambda}, ..., R_N = \frac{\sum_{i=1}^{N} X_i}{\pi\lambda}$$

9. A method for simulating complicated integrals

Suppose we want to evaluate

$$\mu = \int_0^1 k(x)\, dx$$

for a complicated function k. Note that

$$\mu = \mathbf{E}k(U)$$

where $U \sim U(0,1)$.We simulate a large number $U_1, ..., U_n$ of $U(0,1)$ and let $V_i = k(U_i)$. Then the V_i are i.i.d. and therefore by the strong law of large numbers

$$\overline{V} \to \mu$$

If we want an interval estimator, we can treat this as a statistics problem, where we want a confidence interval for μ. Let S^2 be the sample variance of the V_i. Then

$$1 - \alpha = P\left(\mu \in \overline{V} \pm z^{\alpha/2} \frac{S}{\sqrt{n}}\right)$$

Note that by simulating an integral, we can approximate it as close as we like with almost no thinking at all.