The dark matter spectral density from lensed CMB observations

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History of dark matter according to Wikipedia

- First postulated in 1934 by Fritz Zwicky.
- Observing the Coma galaxy cluster.
- Galaxies were spinning too fast given the amount of visible matter.
- Postulated dark matter: invisible, only interacts with regular matter through it’s gravitational influence.

Feel it’s gravity but can’t see it.

- There is a lot of it (∼ 80% of the matter is dark)
- complete understanding has remained elusive
- Large scientific effort probing it’s nature (XENON, CDMS, gravitational lensing...etc)
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Gravitational lensing:

The bending or distortion of photon trajectories by the gravitational influence of intervening matter.

Picture Credit: CREDIT: S. Colombi (IAP), CFHT Team
Gravitational Lensing distorts background images

- Image of distant galaxies
- no intervening matter

No lensing
Gravitational Lensing distorts background images

- Image of distant galaxies
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- same image of distant galaxies
- invisible clump of intervening dark matter

No lensing

Lensing
CMB = deep background image

- Distant and old image of radiation fluctuations during the big bang.
- Provides a wealth of information for cosmology.
- Observations are gravitationally lensed from intervening matter.
No Lensing
Detecting and estimating this lensing is important for two reasons:

- Measuring the distortion gives an indirect measurement of matter in the Universe
- Can use the estimates of gravitational lensing to un-distort the observed CMB

Perfect example of: “One man’s noise is another man’s signal”
Agenda

- Random fields and the CMB (preliminaries)
- Estimating the lensing potential:
  - The Quadratic Estimator
  - Local likelihood techniques
- Simulation
- Challenges
Random fields and the CMB (preliminaries)

- CMB: $\Theta(x)$ for $x \in \mathbb{R}^2$ (flat sky).
- Lensed CMB

$$\Theta(x + \nabla \phi(x))$$

where $\phi(x)$ is the lensing potential.

- Goal: estimate $\phi$
- Both $\phi$ and $\Theta$ are isotropic Gaussian random fields
- Randomness is real (from quantum properties)
  ... not to represent model uncertainty.
Random fields and the CMB (preliminaries)

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Both $\phi$ and $\Theta$ are isotropic Gaussian random fields.

**Definition:**

A *random field* is a collection of random variables, indexed by $\mathbb{R}^d$, all defined on the same probability space: 

$$\{ Z(x) : x \in \mathbb{R}^d \},$$

where $Z(x)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

- $Z$ can be viewed as a random function of $x \in \mathbb{R}^d$: stochastic process or a random image.
Both $\phi$ and $\Theta$ are isotropic Gaussian random fields.

**Definition:** A random field $Z$ is Gaussian if $(Z(x_1), \ldots, Z(x_n))^T$ is a multivariate Gaussian random vector for each $x_1, \ldots, x_n, n$.

- A mean zero Gaussian random field $Z$ is characterized by its covariance function:

  $$C_Z(x, y) \equiv \text{cov}(Z(x), Z(y)).$$
Both $\phi$ and $\Theta$ are isotropic Gaussian random fields

**Definition:**

A random field $Z$ is isotropic if it is statistically invariant to translations and rotations of the coordinates:

$$\{Z(x) : x \in \mathbb{R}^d\} \overset{D}{=} \{Z(R_\theta x + v) : x \in \mathbb{R}^d\}$$

- If $Z$ is an isotropic Gaussian random field then the covariance function depends only on distance:

  $$C_Z(|x - y|) = \text{cov}(Z(x), Z(y))$$

- Physics predicts a model for $C_\phi$ and $C_\Theta$. 
The spectral density

- The spectral density of $\phi$ and $\Theta$ is defined as:

$$C_{\Theta \Theta} \equiv \int_{\mathbb{R}^d} C_\Theta(|x|) e^{-ix \cdot k} \, dx$$

$$C_{\phi \phi} \equiv \int_{\mathbb{R}^d} C_\phi(|x|) e^{-ix \cdot k} \, dx$$

- Characterize the statistical properties of $\phi$ and $\Theta$.
- The spectral density reveals a lot of structure in the CMB.
Isotropic and Gaussian $Z$ has a basis representation with random coefficients:

$$Z(x) = \sum_{k} \xi_{k} e^{ix \cdot k}$$

where $\xi_{k}$ are independent (complex) random variables such that

$$E(\xi_{k}) = 0$$
$$E|\xi_{k}|^2 = \Delta k C_{k}^{ZZ}$$

⇒ if $C_{k}^{ZZ}$ is large then $Z$ has large fluctuations on length scales $2\pi/k$.

⇒ The Fourier transform spatially de-correlates $Z$. 

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Gravitational Lensing.
Spectral density heuristic for the Statistician

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Plots of the covariance functions and spectral densities

Figure: Left: Plot of $C_\phi$ in blue and $C_\Theta$ in green. Notice that 0.1 rads $\approx 5.7^\circ$. Right: Plot of $k^4|C_{|k|}^{\phi\phi}$ in blue and $k^4|C_{|k|}^{\Theta\Theta}$ in green.
Simulation of $\phi$ (left) and $\Theta$ (right)
Fourier transform spatially de-correlates Gravitational Lensing.
Much is known about $C_{|k|}^{\Theta \Theta}$:

Figure: Left: WMAP data. Right: Plot of $\sim k^2 C_{|k|}^{\Theta \Theta}$ along with estimates from WMAP7.
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The Quadratic Estimator

- Derived by Hu and Okamoto (2001-2002)
- Let $\tilde{\Theta}(x) := \Theta(x + \nabla \phi(x))$.
- Taylor expand around $x$
  \[ \tilde{\Theta}(x) \approx \Theta(x) + \nabla \Theta(x) \cdot \nabla \phi(x) \]
- Taking the Fourier transform
  \[ \tilde{\Theta}(k) \approx \Theta(k) + ik\Theta(k) \ast ik\phi(k) \]
  spatially correlated
- Use the spatial correlation in the Fourier domain to estimate $\phi$. 
Some details....

When \( h \neq 0 \)

\[
\text{cov}(\tilde{\Theta}(k + h), \tilde{\Theta}(k)) = E(\tilde{\Theta}(k + h)\tilde{\Theta}(k)^*)
\]
\[
\approx \delta_h C_{|k|}^{\Theta\Theta} + f(k, h) \phi(h)
\]

\[
\therefore E \frac{\tilde{\Theta}(k+h)\tilde{\Theta}(k)^*}{f(k,h)} \approx \phi(h).
\]

\[
\frac{\tilde{\Theta}(k+h)\tilde{\Theta}(k)^*}{f(k,h)} \text{ is a noisy estimate of } \phi(h) \text{ for each } k
\]
\[ \frac{\tilde{\Theta}(k+h)\tilde{\Theta}(k)^*}{f(k,h)} \] is a noisy estimate of \( \phi(h) \) for each \( k \)

Average them to get

\[ \hat{\phi}(h) = \int w(k) \frac{\tilde{\Theta}(k + h)\tilde{\Theta}(k)^*}{f(k, h)} \, dk \]

where \( \int w(k) \, dk = 1 \).

Optimal weights: \( w(k) \propto \frac{|f(k,h)|^2}{C_{\Theta\Theta} |k+h| C_{\Theta\Theta} |k|} \)
Great estimate....with a few problems

- Higher order terms are not insignificant...they introduce bias.
- Has a hard time with:
  - missing pixels
  - partial sky obs
  - nonstationary noise, etc....
- Ignores some interesting features in the polarization.
- On the hunt for alternatives:
Great estimate....with a few problems

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Lensing

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Lensing $\times$ 10
Quadratic Estimator Taylor approx on $\Theta$:

$$\Theta(x + \nabla \phi(x)) \approx \Theta(x) + \nabla \phi(x) \cdot \nabla \Theta(x)$$

Local Taylor approx on $\phi$ (near $x_0$):

$$\phi(x) \approx \phi(x_0) + \nabla \phi(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 \phi(x_0)(x - x_0)$$

call this $q^\phi$

This gives:

$$\Theta(x + \nabla \phi(x)) \approx \Theta(x + \nabla \phi(x_0) + \nabla q^\phi(x))$$

$$\mathcal{D} \Theta(x + \nabla q^\phi(x))$$
\[ \Theta(x + \nabla \phi(x)) \approx \Theta(x + \nabla q^\phi(x)) \]

- \( q^\phi(x) := c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2 \)

- Estimate \( c_1, c_2, c_3 \) locally by fitting \( \Theta(x + \nabla q^\phi(x)) \) locally to obs.

- Stitching together the locally estimated \( q^\phi \) to construct \( \phi \).

Advantages

- Pixel space and local:
  - Nonstationary noise and beams are easy
  - Partial sky observations are easy
  - Missing pixels are easy
- Different Taylor bias than the quadratic estimator
- Easy to include polarization: \((Q(x), U(x))\)
\( \Theta(x + \nabla \phi(x)) \approx \Theta(x + \nabla q^\phi(x)) \)

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Local Likelihood

- Observed CMB:

\[
\Theta^{obs}(x) = \varphi(x) \ast \Theta(x + \nabla \phi(x)) + \sigma N(x)
\]

\[
\approx \text{locally} \underbrace{\varphi(x) \ast \Theta(x + \nabla q^\phi(x)) + \sigma N(x)}_{q^\phi \text{ is quadratic } \Rightarrow \text{this is stationary}}
\]

- The spectral density of \(\varphi(x) \ast \Theta(x + \nabla q^\phi(x))\) is

\[
|\varphi(k)|^2 \frac{C^{\Theta \Theta}_{|M^{-1}k|}}{\det M}
\]

where \(Mx := x + \nabla q^\phi(x)\).
Local Likelihood

- Let $z$ be the vector of observations (from $Q^{obs}, U^{obs}, \Theta^{obs}$) in a small neighborhood.
- Let $\Sigma_{q\phi}$ be the covariance matrix among $z$.
- Then by Gaussianity

\[
\mathcal{L}(q^\phi|z) = -\frac{1}{2} z^\dagger \Sigma_{q\phi}^{-1} z - \frac{1}{2} \ln \det \Sigma_{q\phi}
\]
Maximize the likelihood $\mathcal{L}(q^\phi | z)$

$$\hat{q}^\phi = \arg \max_{q^\phi} \mathcal{L}(q^\phi | z)$$
Stitching together $\hat{q}^{\phi}$ to get $\hat{\nabla} \phi$ and $\hat{\phi}$

Use a gradient fit algorithm: $\nabla F + \text{noise} \xrightarrow{\text{Grad fit}} \hat{F}$. 

\[ \frac{\partial^2 \phi}{\partial x^2} \quad \text{Gradient Fit} \quad \frac{\partial \phi}{\partial x} \quad \text{Gradient Fit} \]

\[ \frac{\partial^2 \phi}{\partial x \partial y} \quad \frac{\partial \phi}{\partial y} \]

\[ \frac{\partial^2 \phi}{\partial y^2} \quad \hat{\phi} \]
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Example: $\Theta^{\text{obs}}$, $Q^{\text{obs}}$, $U^{\text{obs}}$.

Figure: Data: $\Theta^{\text{obs}}$, $Q^{\text{obs}}$, $U^{\text{obs}}$ (17° × 17° patch, 1 $\mu K$-arcmin on $\Theta$, $\sqrt{2}$ $\mu K$-arcmin for the polarization fields, beam FWHM of 0.25 arcmin).
Example: $\Theta^{obs}$, $Q^{obs}$, $U^{obs}$.

Figure: Top: $\Theta^{obs}$, $Q^{obs}$, $U^{obs}$. Bottom: A local estimation neighborhood.
Example: $\Theta^{\text{obs}}, Q^{\text{obs}}, U^{\text{obs}}$.

Figure: Left: estimated potential. Right: true potential.
Example: $\Theta^{\text{obs}}$, $Q^{\text{obs}}$, $U^{\text{obs}}$.

Figure: Left: Estimated $\partial \phi / \partial x$. Right: true $\partial \phi / \partial x$. 

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Local likelihood estimate

- Easily handles sky cuts, nonstationary beams, etc...
- Efficiently aggregates information from CMB and polarization
- Computational tractable and paralizable (in contrast to global likelihood techniques)
- Based on a different Taylor approximation than the quadratic estimator
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What’s the challenge?

- Statistical properties of the estimation error are poorly understood.
- Illustrate with the example of spectral density estimation.

**Convergence $\kappa$**

The convergence $\kappa$ is defined as

$$
\kappa \equiv -\left( \phi_{xx} + \phi_{yy} \right)/2.
$$
Statistical properties of the estimation error are poorly understood.

Illustrate with the example of spectral density estimation

**Convergence $\kappa$**

The convergence $\kappa$ is defined as

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What’s the challenge?

- Statistical properties of the estimation error are poorly understood.
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**Estimated convergence $\kappa$**

The estimated convergence $\kappa$ is

$$\hat{\kappa} \equiv -\left(\hat{\phi}_{xx} + \hat{\phi}_{yy}\right)/2.$$
If $\kappa$ is known

$$\hat{C}_{\ell_0}^{\kappa\kappa} = \frac{\Delta \ell}{\#A_{\ell_0}} \sum_{\ell \in A_{\ell_0}} |\kappa(\ell)|^2$$

If $\kappa$ is estimated by $\hat{\kappa}$ then

$$\hat{C}_{\ell_0}^{\hat{\kappa}\hat{\kappa}} = \frac{\Delta \ell}{\#A_{\ell_0}} \sum_{\ell \in A_{\ell_0}} |\hat{\kappa}(\ell)|^2$$

$\Delta \ell$ denotes the area of the observation grid in $\ell$; $A_{\ell_0}$ denotes a gridded annulus with radius $\ell_0$; $\#A_{\ell_0}$ denotes the number of grid points in $A_{\ell_0}$. 
Spectral Density Estimation: $C_{\ell}^{\kappa\kappa}$

- If $\kappa$ is known

$$\left\langle \hat{C}_{\ell_0}^{\kappa\kappa} \right\rangle_\phi = \frac{\Delta \ell}{\# A_{\ell_0}} \sum_{\ell \in A_{\ell_0}} \left\langle |\kappa(\ell)|^2 \right\rangle_\phi = C_{\ell_0}^{\kappa\kappa}$$

- If $\kappa$ is estimated by $\hat{\kappa}$ then

$$\left\langle \hat{C}_{\ell_0}^{\hat{\kappa}\hat{\kappa}} \right\rangle_\phi = \frac{\Delta \ell}{\# A_{\ell_0}} \sum_{\ell \in A_{\ell_0}} \left\langle |\hat{\kappa}(\ell)|^2 \right\rangle_\phi$$

$$= \frac{\Delta \ell}{\# A_{\ell_0}} \sum_{\ell \in A_{\ell_0}} \left\langle |\kappa(\ell) + N(\ell)|^2 \right\rangle_\phi$$

$$= C_{\ell_0}^{\kappa\kappa} + C_{\ell_0}^{NN}, \text{ if } \kappa \perp N$$

(additive bias)
The bias is large

Figure: 20 simulations of lensed CMB intensity: Noise $sd = 2 \ \mu K$, Beam FWHM = 4, $10^\circ \times 10^\circ$ patch of sky, Fixed potential over different noise and CMB realizations.
With lensing

Without lensing

\[ \hat{C}_\ell^{\hat{\kappa}\hat{\kappa}} \approx C_\ell^{\kappa\kappa} + C_\ell^{NN} \]

additive bias

\[ \approx C_\ell^{NN} \]

additive bias
Subtract the estimated bias

\[ \hat{C}_\ell^{\kappa\kappa} - \hat{C}_\ell^{NN} \text{ vrs } \hat{C}_\ell^{\kappa\kappa} \]

True Spectrum
Est using the true kappa
Bias correction

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The challenge is to understand $C_{\ell}^{NN}$ or $\hat{C}_{\ell}^{NN}$

- $\hat{C}_{\ell}^{NN}$ doesn’t contain information on the Taylor truncation bias
- Simulations for $\hat{C}_{\ell}^{NN}$ require a model.
- Need a theoretical understanding of $C_{\ell}^{NN}$ for the local likelihood method to be scientifically useful.

$\hat{C}_{\ell}^{KK} - C_{\ell}^{NN}$ vrs $\hat{C}_{\ell}^{KK}$
Thanks!!
Example: $\Theta^{\text{obs}}$ only.

**Figure:** noise level $1/2 \, \mu K$ arcmin with a beam FWHM 1 armin observed on $10^\circ \times 10^\circ$ patch of the sky. The quadratic estimator uses a periodic boundary $17^\circ \times 17^\circ$ patch.
Example: $\Theta^{\text{obs}}$ only.

Figure: noise level $1/2 \, \mu K \text{ arcmin}$ with a beam FWHM 1 arcmin observed on $10^\circ \times 10^\circ$ patch of the sky. The quadratic estimator uses a periodic boundary $17^\circ \times 17^\circ$ patch.
Example: $\Theta^{\text{obs}}$ only.

**Figure:** Local likelihood estimated potential (left) true potential (middle) and the quadratic estimated potential (right). Note: the likelihood estimate is based on 100 square degrees of sky and the quadratic estimate is based on 289 square degrees of the sky.
Example: $\Theta^{\text{obs}}$ only.

Figure: Local likelihood estimated $\partial \phi / \partial x$ (left) true $\partial \phi / \partial x$ (middle) and the quadratic estimated $\partial \phi / \partial x$ (right).
Figure: Local likelihood estimated $\partial \phi / \partial y$ (left) true $\partial \phi / \partial y$ (middle) and the quadratic estimated $\partial \phi / \partial y$ (right).
Behavior of $\hat{q}^\phi$

- $\hat{q}^\phi$ behaves like a local quadratic fit to $\phi$.

\[
\sum_k \xi_k e^{ikx} = \sum_{k: \frac{2\pi}{k} < \delta} \xi_k e^{ikx} + \sum_{k: \frac{2\pi}{k} > 2\delta} \xi_k e^{ikx}
\]

wavelengths $< \delta$

wavelengths $> 2\delta$
$\hat{q}^\phi$ estimates a low pass filter of $\phi$.

The approximate filter: \[ \left\{ 1 \land \left[ 2 - \frac{\delta}{\pi} |k| \right]^+ \right\} \]

In pixel space:

$$\hat{q}^\phi(x) \approx \sum_k \left\{ 1 \land \left[ 2 - \frac{\delta}{\pi} |k| \right]^+ \right\} \xi_k e^{i x \cdot k}$$

$$\approx \int \frac{dk}{2\pi} e^{i x \cdot k} \left\{ 1 \land \left[ 2 - \frac{\delta}{\pi} |k| \right]^+ \right\} \phi(k)$$

$$=: \phi^{lp}(x)$$
Example: $\Theta^{\text{obs}}, Q^{\text{obs}}, U^{\text{obs}}$.

Figure: Estimated $\frac{\partial^2 \hat{\phi}}{\partial x \partial y}$ (top) low pass filter of the truth $\frac{\partial^2 \phi^{\text{lp}}}{\partial x \partial y}$ (bottom left) true $\frac{\partial^2 \phi}{\partial x \partial y}$ (bottom right).