Estimation of moments on azimuthally symmetric patches on the sphere by means of fast convolution

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Abstract

In order to study the statistical properties of large data sets, fast and reliable methods for the estimation of basic statistical quantities, such as moments of the data, are required. We present a method for the estimation of moments on azimuthally symmetric patches defined for data pixelized on the sphere by means of fast convolution. As an example application, we show the results of a search in the WMAP CMB sky maps for regions with anomalous values of the variance, skewness or kurtosis as estimated on a set of concentric rings.

Computation of moments by fast convolution on the sphere

The computation of moments on azimuthally symmetric patches on the sphere can be viewed as the convolution of the data, taken to the appropriate power, with an azimuthally symmetric beam that describes the geometry of the patch. To make this statement more clear, let us consider the example of the computation of variance on the pixelized map $\Delta T$, for which some regions are excluded by application of the mask $M_r$. For a region centered on, but not necessarily including, pixel $i$ an estimator of the variance $\widehat{\chi}_{\text{pix}}^2$ is given by

$$\widehat{\chi}_{\text{pix}}^2 = \left( \sum_{j} \Delta T_i M_{ij} \right)^2 / \left( \sum_{j} \Delta T_i M_{ij}^2 \right) .$$

Here, $\Delta T_i$ denotes the profile of the azimuthally symmetric patch used for estimation of the variance. In the case of our studies of CMB maps presented here, this corresponds to a ring with radius of $r$ and width $\Delta r$ such that $\Delta r_i = \left\{ \begin{array}{ll} \frac{1}{2} - \text{arccos}(\hat{a}_i \cdot \hat{k}) & \text{if } r \leq \text{arccos}(\hat{a}_i \cdot \hat{k}) + \Delta r_i \Delta \theta \text{ otherwise} \end{array} \right.$ (2)

The last term in (1) corresponds to a correction for the bias introduced by the noise variance $\sigma^2$. Computation of this estimator is achieved by direct summation over all unmasked pixels $j$ in a given patch centered on pixel $i$ for all possible centers of the patch, and scaled as $O(N^{1/2})$, where $N_{\text{pix}}$ is the number of pixels in the map. However, this sum is nothing other than the convolution of the masked map $\Delta T_i M_{ij}$ with beam $B_{\Delta T_i}$. Therefore, it can be performed efficiently by decomposing the data in the basis of spherical harmonics $\chi_{ij}(\hat{l})$ functions. Then, for example, the sum of the terms in $\Delta T_i M_{ij}$ is given by

$$\sum_{j} \Delta T_i M_{ij} = \sum_{n} \phi_{n} \chi_{ij}(\hat{n}) ,$$

where $\phi_n$ and $B_{\Delta T_i}$ are the spherical harmonic coefficients of the masked map $\Delta T_i$ and patch, respectively. In the case of azimuthally symmetric patches, we have implicitly utilized the fact that the coefficients $\phi_n$ can depend only on the multipole order $\ell$. Therefore, the complexity of the algorithm for the computation of the variance can be reduced to $O(N_{\text{pix}} \log N_{\text{pix}})$ operations. This algorithm can also be extended to higher order moments, such as the skewness or kurtosis.

Search for the anomalous regions in the WMAP maps

We employ this technique to search for regions of the 7-year WMAP data (Jarosik et al., 2011) which exhibit anomalous variance, skewness or kurtosis.

In Fig. 1 we show an example of the variances computed for rings centered at the Galactic coordinates $(b, l) = (90^\circ, 37^\circ)$. This point recently drew special attention in the CMB community (Hajian, 2010; Moss, Scott, & Zibin, 2010; Wehus & Eriksen, 2010). Our method allows us to very efficiently estimate variances for a more complete range of ring radii. As we can see, for many rings the observed variance is lower than the corresponding mean variance estimated on the basis of 1000 Monte Carlo (MC) simulations for the standard $\Lambda$CDM model, that include noise contributions appropriate to the WMAP frequencies of interest. This may be connected to various large angular scale anomalies observed in the WMAP data (Copi et al., 2004; Eriksen et al., 2004; Hansen et al., 2004; Schwarz et al., 2004).

In order to quantify the significance of the overall fluctuations of the measured variances on concentric rings, we used the $\chi^2$ statistic defined as

$$\chi^2 = \sum_{r, \ell} \left( \frac{\hat{\sigma}_{\nu r}^2}{\sigma_{\nu r}^2} - 1 \right) C_{\nu r}^{-1}(\hat{n}) \right) ,$$

where $\nu_{r, \ell}$ and $C_{\nu r}$ are mean variance and covariance matrix of the variances for a given pixel $i$, respectively. In Fig. 5 we show the $\chi^2$-map for rings with radii in the range $[9^\circ, 13^\circ]$. As we can see, there are a few regions with higher values of the $\chi^2$ statistic. However, comparison of the distribution of the $\chi^2$-map with MC simulations (see Fig. 6) does not support any significant inconsistency of these regions with the expectations for the standard $\Lambda$CDM model.

Summary

We have presented an efficient algorithm for the estimation of moments on azimuthally symmetric patches on the sphere by means of fast convolution. The low complexity of the algorithm makes it well suited for the search for azimuthally symmetric patches on the sphere by means of fast convolution. Preliminary results of the application of this method to the WMAP data reveal a few interesting regions on the sky, but general consistency with the currently preferred standard $\Lambda$CDM model.

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References