Introduction to Bayesian Inference
Lecture 1: Fundamentals

Tom Loredo
Dept. of Astronomy, Cornell University
http://www.astro.cornell.edu/staff/loredo/bayes/

CASt Summer School — 10 June 2011
Frequentist Estimation of a Normal Mean

Suppose we have a sample of \( N = 5 \) values \( x_i \),

\[ x_i \sim N(\mu, 1) \]

We want to estimate \( \mu \), including some quantification of uncertainty in the estimate: an interval with a probability attached.

Frequentist approaches: method of moments, BLUE, least-squares/\( \chi^2 \), maximum likelihood

Focus on likelihood (equivalent to \( \chi^2 \) here); this is closest to Bayes.

\[
\mathcal{L}(\mu) = p(\{x_i\} | \mu) = \prod_i \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}}; \quad \sigma = 1 \\
\propto e^{-\chi^2(\mu)/2}
\]

Estimate \( \mu \) from maximum likelihood (minimum \( \chi^2 \)).
Define an interval and its coverage frequency from the \( \mathcal{L}(\mu) \) curve.
Construct an Interval Procedure for Known $\mu$

Likelihoods for 3 simulated data sets, $\mu = 0$
Likelihoods for 100 simulated data sets, $\mu = 0$
Explore Dependence on $\mu$
Likelihoods for 100 simulated data sets, $\mu = 3$

Sample Space

Parameter Space

Luckily the $\Delta \log \mathcal{L}$ distribution is the same!
Report the green region, with coverage as calculated for ensemble of hypothetical data (red region, previous slide).
Bayesian Approach

Normalize the likelihood for the observed sample; report the region that includes 68.3% of the normalized likelihood.
When They’ll Differ

Both approaches report $\mu \in [\bar{x} - \sigma/\sqrt{N}, \bar{x} + \sigma/\sqrt{N}]$, and assign 68.3% to this interval (with different meanings).

This matching is a *coincidence*!

When might results differ? ($F = \text{frequentist}, \ B = \text{Bayes}$)

- If $F$ procedure doesn’t use likelihood directly
- If $F$ procedure properties depend on params (nonlinear models, pivotal quantities)
- If $F$ properties depend on likelihood shape (conditional inference, ancillary statistics, recognizable subsets)
- If there are extra uninteresting parameters (nuisance parameters, corrected profile likelihood, conditional inference)
- If $B$ uses important prior information

Also, for a different task—comparison of parametric models—the approaches are qualitatively different (significance tests & info criteria vs. Bayes factors)
Bayesian and Frequentist Inference

Brad Efron, ASA President (2005)

The 250-year debate between Bayesians and frequentists is unusual among philosophical arguments in actually having important practical consequences... The physicists I talked with were really bothered by our 250 year old Bayesian-frequentist argument. Basically there’s only one way of doing physics but there seems to be at least two ways to do statistics, and they don’t always give the same answers....

Broadly speaking, Bayesian statistics dominated 19th Century statistical practice while the 20th Century was more frequentist. What’s going to happen in the 21st Century?... I strongly suspect that statistics is in for a burst of new theory and methodology, and that this burst will feature a combination of Bayesian and frequentist reasoning....
Pragmatists might argue that good statisticians can get sensible answers under Bayes or frequentist paradigms; indeed maybe two philosophies are better than one, since they provide more tools for the statistician’s toolkit... I am discomforted by this “inferential schizophrenia.” Since the Bayesian (B) and frequentist (F) philosophies can differ even on simple problems, at some point decisions seem needed as to which is right. I believe our credibility as statisticians is undermined when we cannot agree on the fundamentals of our subject... An assessment of strengths and weaknesses of the frequentist and Bayes systems of inference suggests that calibrated Bayes... captures the strengths of both approaches and provides a roadmap for future advances.
Lecture 1: Fundamentals

1. The big picture

2. Foundations: Logic & probability theory

3. Probability theory for data analysis: Three theorems

4. Inference with parametric models
   - Parameter Estimation
   - Model Uncertainty
Bayesian Fundamentals

1. The big picture

2. Foundations: Logic & probability theory

3. Probability theory for data analysis: Three theorems

4. Inference with parametric models
   - Parameter Estimation
   - Model Uncertainty
**Scientific Method**

*Science is more than a body of knowledge; it is a way of thinking.*
*The method of science, as stodgy and grumpy as it may seem, is far more important than the findings of science.*

—Carl Sagan

Scientists *argue!*

Argument ≡ Collection of statements comprising an act of reasoning from *premises* to a *conclusion*

A key goal of science: Explain or predict *quantitative measurements* (data!)

*Data analysis:* Constructing and appraising arguments that reason from data to interesting scientific conclusions (explanations, predictions)
The Role of Data

*Data do not speak for themselves!*

We don't just *tabulate* data, we *analyze* data.

We gather data so they may speak for or against existing hypotheses, and guide the formation of new hypotheses.

A key role of data in science is to be among the premises in scientific arguments.
Data Analysis
Building & Appraising Arguments Using Data

Modes of Data Analysis

*Statistical inference* is but one of several interacting modes of analyzing data.
Bayesian Statistical Inference

- A different approach to *all* statistical inference problems (i.e., not just another method in the list: BLUE, maximum likelihood, $\chi^2$ testing, ANOVA, survival analysis . . . )

- Foundation: Use probability theory to quantify the strength of arguments (i.e., a more abstract view than restricting PT to describe variability in repeated “random” experiments)

- Focuses on *deriving consequences of modeling assumptions* rather than *devising and calibrating procedures*
Frequentist vs. Bayesian Statements

“I find conclusion $C$ based on data $D_{\text{obs}}$ . . . ”

Frequentist assessment

“It was found with a procedure that’s right 95% of the time over the set $\{D_{\text{hyp}}\}$ that includes $D_{\text{obs}}$.”

Probabilities are properties of procedures, not of particular results.

Bayesian assessment

“The strength of the chain of reasoning from $D_{\text{obs}}$ to $C$ is 0.95, on a scale where 1 = certainty.”

Probabilities are properties of specific results.

Long-run performance must be separately evaluated (and is typically good by frequentist criteria).
Bayesian Fundamentals

1. The big picture

2. Foundations: Logic & probability theory

3. Probability theory for data analysis: Three theorems

4. Inference with parametric models
   Parameter Estimation
   Model Uncertainty
Logic—Some Essentials

“Logic can be defined as the analysis and appraisal of arguments”
—Gensler, *Intro to Logic*

Build arguments with propositions and logical operators/connectives:

- **Propositions:** Statements that may be true or false
  - \( P \): Universe can be modeled with ΛCDM
  - \( A \): \( \Omega_{\text{tot}} \in [0.9, 1.1] \)
  - \( B \): \( \Omega_{\Lambda} \) is not 0
  - \( \overline{B} \): “not \( B \),” i.e., \( \Omega_{\Lambda} = 0 \)

- **Connectives:**
  - \( A \land B \): \( A \) and \( B \) are both true
  - \( A \lor B \): \( A \) or \( B \) is true, or both are
Arguments

Argument: Assertion that an hypothesized conclusion, $H$, follows from premises, $\mathcal{P} = \{A, B, C, \ldots\}$ (take “,” = “and”)

Notation:

$$H|\mathcal{P} : \quad \text{Premises } \mathcal{P} \text{ imply } H$$

$H$ may be deduced from $\mathcal{P}$

$H$ follows from $\mathcal{P}$

$H$ is true given that $\mathcal{P}$ is true

Arguments are (compound) propositions.

Central role of arguments $\rightarrow$ special terminology for true/false:

- A true argument is valid
- A false argument is invalid or fallacious
Valid vs. Sound Arguments

Content vs. form

- An argument is *factually correct* iff all of its *premises are true* (it has “good content”).
- An argument is *valid* iff its conclusion *follows from* its premises (it has “good form”).
- An argument is *sound* iff it is both *factually correct and valid* (it has good form and content).

Deductive logic (and probability theory) addresses *validity*.

We want to make *sound* arguments. There is no formal approach for addressing factual correctness → there is always a subjective element to an argument.
Factual Correctness

**Passing the buck**

Although logic can teach us something about validity and invalidity, it can teach us very little about factual correctness. The question of the truth or falsity of individual statements is primarily the subject matter of the sciences.

— Hardegree, *Symbolic Logic*

**An open issue**

To test the truth or falsehood of premisses is the task of science. . . . But as a matter of fact we are interested in, and must often depend upon, the correctness of arguments whose premisses are not known to be true.

— Copi, *Introduction to Logic*
Premises

- **Facts** — Things known to be true, e.g. *observed data*

- “Obvious” assumptions — Axioms, postulates, e.g., Euclid’s first 4 postulates (line segment b/t 2 points; congruency of right angles . . . )

- “Reasonable” or “working” assumptions — E.g., Euclid’s fifth postulate (parallel lines)

- Desperate presumption!

- Conclusions from other arguments

Premises define a fixed *context* in which arguments may be assessed.

Premises are considered “given” — if only for the sake of the argument!
Deductive and Inductive Inference

**Deduction—Syllogism as prototype**

Premise 1: A implies H  
Premise 2: A is true  
Deduction: \( \therefore H \) is true  
\( H \mid \mathcal{P} \) is valid

**Induction—Analogy as prototype**

Premise 1: A, B, C, D, E all share properties \( x, y, z \)  
Premise 2: F has properties \( x, y \)  
Induction: F has property z  
“F has z” \( \mid \mathcal{P} \) is not strictly valid, but may still be rational  
(likely, plausible, probable); some such arguments are stronger than others

*Boolean algebra* (and/or/not over \( \{0, 1\} \)) quantifies deduction.

*Bayesian probability theory* (and/or/not over \([0, 1]\)) generalizes this to quantify the strength of inductive arguments.
Deductive Logic

Assess arguments by decomposing them into parts via connectives, and assessing the parts:

**Validity of \( A \land B \mid \mathcal{P} \)**

<table>
<thead>
<tr>
<th>( B \mid \mathcal{P} )</th>
<th>( A \mid \mathcal{P} )</th>
<th>( \overline{A} \mid \mathcal{P} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>valid</td>
<td>invalid</td>
<td>invalid</td>
</tr>
<tr>
<td>invalid</td>
<td>invalid</td>
<td>invalid</td>
</tr>
</tbody>
</table>

**Validity of \( A \lor B \mid \mathcal{P} \)**

<table>
<thead>
<tr>
<th>( B \mid \mathcal{P} )</th>
<th>( A \mid \mathcal{P} )</th>
<th>( \overline{A} \mid \mathcal{P} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>valid</td>
<td>valid</td>
<td>valid</td>
</tr>
<tr>
<td>valid</td>
<td>invalid</td>
<td>invalid</td>
</tr>
</tbody>
</table>
**Representing Deduction With \{0, 1\} Algebra**

$V(H|\mathcal{P}) \equiv$ Validity of argument $H|\mathcal{P}$:

- $V = 0 \rightarrow$ Argument is *invalid*
- $V = 1 \rightarrow$ Argument is *valid*

Then deduction can be reduced to integer multiplication and addition over \{0, 1\} (as in a computer):

\[
\begin{align*}
V(A \land B|\mathcal{P}) &= V(A|\mathcal{P}) V(B|\mathcal{P}) \\
V(A \lor B|\mathcal{P}) &= V(A|\mathcal{P}) + V(B|\mathcal{P}) - V(A \land B|\mathcal{P}) \\
V(\overline{A}|\mathcal{P}) &= 1 - V(A|\mathcal{P})
\end{align*}
\]
Representing Induction With $[0, 1]$ Algebra

$P(H|\mathcal{P}) \equiv$ strength of argument $H|\mathcal{P}$

$P = 0 \rightarrow$ Argument is invalid; premises imply $\overline{H}$

$= 1 \rightarrow$ Argument is valid

$\in (0, 1) \rightarrow$ Degree of deducibility

Mathematical model for induction

‘AND’ (product rule): $P(A \land B|\mathcal{P}) = P(A|\mathcal{P})P(B|A \land \mathcal{P})$

$= P(B|\mathcal{P})P(A|B \land \mathcal{P})$

‘OR’ (sum rule): $P(A \lor B|\mathcal{P}) = P(A|\mathcal{P}) + P(B|\mathcal{P})$

$- P(A \land B|\mathcal{P})$

‘NOT’: $P(\overline{A}|\mathcal{P}) = 1 - P(A|\mathcal{P})$
The Product Rule

We simply promoted the $V$ algebra to real numbers; the only thing changed is part of the product rule:

\[
V(A \land B | \mathcal{P}) = V(A | \mathcal{P}) V(B | \mathcal{P})
\]
\[
P(A \land B | \mathcal{P}) = P(A | \mathcal{P}) P(B | A, \mathcal{P})
\]

Suppose $A$ implies $B$ (i.e., $B | A, \mathcal{P}$ is valid). Then we don’t expect $P(A \land B | \mathcal{P})$ to differ from $P(A | \mathcal{P})$.

In particular, $P(A \land A | \mathcal{P})$ must equal $P(A | \mathcal{P})$!

Such qualitative reasoning satisfied early probabilists that the sum and product rules were worth considering as axioms for a theory of quantified induction.
Today many different formal lines of argument derive induction-as-probability from various simple and appealing requirements:

- Consistency with logic + internal consistency (Cox; Jaynes)
- “Coherence”/optimal betting (Ramsey; DeFinetti; Wald; Savage)
- Algorithmic information theory (Rissanen; Wallace & Freeman)
- Optimal information processing (Zellner)
- Avoiding problems with frequentist methods:
  - Avoiding recognizable subsets (Cornfield)
  - Avoiding stopping rule problems → likelihood principle (Birnbaum; Berger & Wolpert)
Interpreting Bayesian Probabilities

If we like there is no harm in saying that a probability expresses a degree of reasonable belief. . . . ‘Degree of confirmation’ has been used by Carnap, and possibly avoids some confusion. But whatever verbal expression we use to try to convey the primitive idea, this expression cannot amount to a definition. Essentially the notion can only be described by reference to instances where it is used. It is intended to express a kind of relation between data and consequence that habitually arises in science and in everyday life, and the reader should be able to recognize the relation from examples of the circumstances when it arises.

— Sir Harold Jeffreys, *Scientific Inference*
More On Interpretation

Physics uses words drawn from ordinary language—mass, weight, momentum, force, temperature, heat, etc.—but their technical meaning is more abstract than their colloquial meaning. We can map between the colloquial and abstract meanings associated with specific values by using specific instances as “calibrators.”

A Thermal Analogy

<table>
<thead>
<tr>
<th>Intuitive notion</th>
<th>Quantification</th>
<th>Calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hot, cold</td>
<td>Temperature, $T$</td>
<td>Cold as ice $= 273K$ \ Boiling hot $= 373K$</td>
</tr>
<tr>
<td>uncertainty</td>
<td>Probability, $P$</td>
<td>Certainty $= 0, 1$ \ $p = 1/36$: plausible as “snake’s eyes” \ $p = 1/1024$: plausible as 10 heads</td>
</tr>
</tbody>
</table>
Bayesian

Probability *quantifies uncertainty* in an inductive inference. \( p(x) \) describes how *probability* is distributed over the possible values \( x \) might have taken in the single case before us:

![Bayesian Diagram](image)

Frequentist

Probabilities are always (limiting) rates/proportions/frequencies that *quantify variability* in a sequence of trials. \( p(x) \) describes how the *values of \( x \) would be distributed* among infinitely many trials:

![Frequentist Diagram](image)
Bayesian Fundamentals

1. The big picture

2. foundations: Logic & probability theory

3. Probability theory for data analysis: Three theorems

4. Inference with parametric models
   - Parameter Estimation
   - Model Uncertainty
Arguments Relating Hypotheses, Data, and Models

We seek to appraise scientific hypotheses in light of observed data and modeling assumptions.

Consider the data and modeling assumptions to be the premises of an argument with each of various hypotheses, $H_i$, as conclusions: $H_i|D_{\text{obs}}, I$. ($I = \text{“background information,” everything deemed relevant besides the observed data}$)

$P(H_i|D_{\text{obs}}, I)$ measures the degree to which $(D_{\text{obs}}, I)$ allow one to deduce $H_i$. It provides an ordering among arguments for various $H_i$ that share common premises.

Probability theory tells us how to analyze and appraise the argument, i.e., how to calculate $P(H_i|D_{\text{obs}}, I)$ from simpler, hopefully more accessible probabilities.
The Bayesian Recipe

Assess hypotheses by calculating their probabilities $p(H_i|\ldots)$ conditional on known and/or presumed information using the rules of probability theory.

*Probability Theory Axioms:*

‘OR’ (sum rule): \[ P(H_1 \lor H_2|I) = P(H_1|I) + P(H_2|I) - P(H_1, H_2|I) \]

‘AND’ (product rule): \[ P(H_1, D|I) = P(H_1|I) P(D|H_1, I) = P(D|I) P(H_1|D, I) \]

‘NOT’: \[ P(\overline{H_1}|I) = 1 - P(H_1|I) \]
Three Important Theorems

Bayes’s Theorem (BT)

Consider \( P(H_i, D_{obs}|I) \) using the product rule:

\[
P(H_i, D_{obs}|I) = P(H_i|I) P(D_{obs}|H_i, I) \]
\[
= P(D_{obs}|I) P(H_i|D_{obs}, I)
\]

Solve for the posterior probability:

\[
P(H_i|D_{obs}, I) = P(H_i|I) \frac{P(D_{obs}|H_i, I)}{P(D_{obs}|I)}
\]

Theorem holds for any propositions, but for hypotheses & data the factors have names:

posterior \( \propto \) prior \( \times \) likelihood

norm. const. \( P(D_{obs}|I) = \) prior predictive
Law of Total Probability (LTP)

Consider exclusive, exhaustive \{B_i\} (I asserts one of them must be true),

$$\sum_i P(A, B_i | I) = \sum_i P(B_i | A, I)P(A | I) = P(A | I)$$

$$= \sum_i P(B_i | I)P(A | B_i, I)$$

If we do not see how to get \(P(A | I)\) directly, we can find a set \{B_i\} and use it as a “basis”—extend the conversation:

$$P(A | I) = \sum_i P(B_i | I)P(A | B_i, I)$$

If our problem already has \(B_i\) in it, we can use LTP to get \(P(A | I)\) from the joint probabilities—marginalization:

$$P(A | I) = \sum_i P(A, B_i | I)$$
Example: Take \( A = D_{\text{obs}}, B_i = H_i \); then

\[
P(D_{\text{obs}}|I) = \sum_i P(D_{\text{obs}}, H_i|I)
\]

\[
= \sum_i P(H_i|I)P(D_{\text{obs}}|H_i, I)
\]

prior predictive for \( D_{\text{obs}} \) = Average likelihood for \( H_i \)
(a.k.a. marginal likelihood)

**Normalization**

For exclusive, exhaustive \( H_i \),

\[
\sum_i P(H_i|\cdots) = 1
\]
Well-Posed Problems

The rules express desired probabilities in terms of other probabilities.

To get a numerical value *out*, at some point we have to put numerical values *in*.

*Direct probabilities* are probabilities with numerical values determined directly by premises (via modeling assumptions, symmetry arguments, previous calculations, desperate presumption . . . ).

An inference problem is *well posed* only if all the needed probabilities are assignable based on the premises. We may need to add new assumptions as we see what needs to be assigned. We may not be entirely comfortable with what we need to assume! (Remember Euclid’s fifth postulate!)

Should explore how results depend on uncomfortable assumptions (“robustness”).
Visualizing Bayesian Inference

Simplest case: Binary classification

- 2 hypotheses: \( \{ H, C \} \)
- 2 possible data values: \( \{-, +\} \)

Concrete example: You test positive (+) for a medical condition. Do you have the condition (C) or not (H, “healthy”)?

- Prior: Prevalence of the condition in your population is 1%
- Likelihood:
  - Test is 90% accurate if you have the condition:
    \[ P(+|C, I) = 0.9 \]  (“sensitivity”)
  - Test is 95% accurate if you are healthy:
    \[ P(-|H, I) = 0.95 \]  (“specificity”)

*Numbers roughly correspond to breast cancer in asymptomatic women aged 40–50, and mammography screening*

[Gigerenzer, *Calculated Risks* (2002)]
$$P(C|+, I) = \frac{0.009}{0.0585} \approx 0.15$$

*Not really a tree; really a graph or part of a lattice*
Integers are easier than reals!
Create a large ensemble of cases so ratios of counts approximate the probabilities.

Of the 59 cases with positive test results, only 9 have the condition. The prevalence is so low that when there is a positive result, it’s more likely to have been a mistake than accurate.
Bayesian Fundamentals

1. The big picture

2. Foundations: Logic & probability theory

3. Probability theory for data analysis: Three theorems

4. Inference with parametric models
   - Parameter Estimation
   - Model Uncertainty
Inference With Parametric Models

Models $M_i$ ($i = 1$ to $N$), each with parameters $\theta_i$, each imply a **sampling dist’n** (conditional predictive dist’n for possible data):

$$p(D|\theta_i, M_i)$$

The $\theta_i$ dependence when we fix attention on the **observed** data is the **likelihood function**:

$$\mathcal{L}_i(\theta_i) \equiv p(D_{obs}|\theta_i, M_i)$$

We may be uncertain about $i$ (model uncertainty) or $\theta_i$ (parameter uncertainty).

*Henceforth we will only consider the actually observed data, so we drop the cumbersome subscript: $D = D_{obs}$.***
Three Classes of Problems

Parameter Estimation

Premise = choice of model (pick specific $i$)

$\rightarrow$ What can we say about $\theta_i$?

Model Assessment

- Model comparison: Premise $= \{M_i\}$
  $\rightarrow$ What can we say about $i$?

- Model adequacy/GoF: Premise $= M_1 \lor$ “all” alternatives
  $\rightarrow$ Is $M_1$ adequate?

Model Averaging

Models share some common params: $\theta_i = \{\phi, \eta_i\}$

$\rightarrow$ What can we say about $\phi$ w/o committing to one model?
(Examples: systematic error, prediction)
**Problem statement**

\( I = \text{Model } M \text{ with parameters } \theta (+ \text{ any add'l info}) \)

\( H_i = \text{statements about } \theta; \text{ e.g. } "\theta \in [2.5, 3.5]," \text{ or } "\theta > 0" \)

Probability for any such statement can be found using a *probability density function* (PDF) for \( \theta \):

\[
P(\theta \in [\theta, \theta + d\theta]| \cdots) = f(\theta)d\theta
= p(\theta| \cdots)d\theta
\]

**Posterior probability density**

\[
p(\theta|D, M) = \frac{p(\theta|M) \mathcal{L}(\theta)}{\int d\theta p(\theta|M) \mathcal{L}(\theta)}
\]
**Summaries of posterior**

- "Best fit" values:
  - Mode, $\hat{\theta}$, maximizes $p(\theta|D, M)$
  - Posterior mean, $\langle \theta \rangle = \int d\theta \theta p(\theta|D, M)$

- Uncertainties:
  - Credible region $\Delta$ of probability $C$:
    $C = P(\theta \in \Delta|D, M) = \int_{\Delta} d\theta p(\theta|D, M)$
    **Highest Posterior Density (HPD) region has** $p(\theta|D, M)$ higher inside than outside
  - Posterior standard deviation, variance, covariances

- Marginal distributions
  - Interesting parameters $\phi$, nuisance parameters $\eta$
  - Marginal dist’n for $\phi$: $p(\phi|D, M) = \int d\eta p(\phi, \eta|D, M)$
Nuisance Parameters and Marginalization

To model most data, we need to introduce parameters besides those of ultimate interest: *nuisance parameters*.

**Example**

We have data from measuring a rate \( r = s + b \) that is a sum of an interesting signal \( s \) and a background \( b \).

We have additional data just about \( b \).

What do the data tell us about \( s \)?
Marginal posterior distribution

To summarize implications for \( s \), accounting for \( b \) uncertainty, **marginalize**:

\[
\begin{align*}
p(s|D, M) &= \int db \, p(s, b|D, M) \\
&\propto p(s|M) \int db \, p(b|s) \, \mathcal{L}(s, b) \\
&= p(s|M) \mathcal{L}_m(s)
\end{align*}
\]

with \( \mathcal{L}_m(s) \) the **marginal likelihood for** \( s \):

\[
\mathcal{L}_m(s) \equiv \int db \, p(b|s) \, \mathcal{L}(s, b)
\]
Marginalization vs. Profiling

*For insight:* Suppose the prior is broad compared to the likelihood → for a fixed $s$, we can accurately estimate $b$ with max likelihood $\hat{b}_s$, with small uncertainty $\delta b_s$.

$$L_m(s) \equiv \int db \ p(b|s) \mathcal{L}(s, b)$$

$$\approx p(\hat{b}_s|s) \mathcal{L}(s, \hat{b}_s) \delta b_s$$

Profile likelihood $\mathcal{L}_p(s) \equiv \mathcal{L}(s, \hat{b}_s)$ gets weighted by a *parameter space volume factor*

E.g., Gaussians: $\hat{s} = \hat{r} - \hat{b}, \quad \sigma_s^2 = \sigma_r^2 + \sigma_b^2$

Background *subtraction* is a special case of background *marginalization*. 
Bivariate normals: $\mathcal{L}_m \propto \mathcal{L}_p$

$\delta b_s$ is const. vs. $s$

$\Rightarrow \mathcal{L}_m \propto \mathcal{L}_p$
Flared/skewed/banana-shaped: $\mathcal{L}_m$ and $\mathcal{L}_p$ differ

General result: For a linear (in params) model sampled with Gaussian noise, and flat priors, $\mathcal{L}_m \propto \mathcal{L}_p$. Otherwise, they will likely differ.

In measurement error problems (future lecture!) the difference can be dramatic.
Many Roles for Marginalization

Eliminate nuisance parameters

\[ p(\phi|D, M) = \int d\eta \ p(\phi, \eta|D, M) \]

Propagate uncertainty

Model has parameters \( \theta \); what can we infer about \( F = f(\theta) \)?

\[
\begin{align*}
p(F|D, M) &= \int d\theta \ p(F, \theta|D, M) = \int d\theta \ p(\theta|D, M) p(F|\theta, M) \\
&= \int d\theta \ p(\theta|D, M) \delta[F - f(\theta)] \quad \text{[single-valued case]}
\end{align*}
\]

Prediction

Given a model with parameters \( \theta \) and present data \( D \), predict future data \( D' \) (e.g., for experimental design):

\[
\begin{align*}
p(D'|D, M) &= \int d\theta \ p(D', \theta|D, M) = \int d\theta \ p(\theta|D, M) p(D'|\theta, M)
\end{align*}
\]

Model comparison...
Model Comparison

Problem statement

\[ I = (M_1 \lor M_2 \lor \ldots) \] — Specify a set of models.

\[ H_i = M_i \] — Hypothesis chooses a model.

Posterior probability for a model

\[
p(M_i|D, I) = \frac{p(M_i|I)p(D|M_i, I)}{p(D|I)} \propto p(M_i|I)\mathcal{L}(M_i)
\]

But \( \mathcal{L}(M_i) = p(D|M_i) = \int d\theta_i p(\theta_i|M_i)p(D|\theta_i, M_i). \)

Likelihood for model = Average likelihood for its parameters

\[
\mathcal{L}(M_i) = \langle \mathcal{L}(\theta_i) \rangle
\]

Varied terminology: Prior predictive = Average likelihood = Global likelihood = Marginal likelihood = (Weight of) Evidence for model
Odds and Bayes factors

A ratio of probabilities for two propositions using the same premises is called the **odds** favoring one over the other:

\[
O_{ij} \equiv \frac{p(M_i|D,I)}{p(M_j|D,I)} = \frac{p(M_i|I)}{p(M_j|I)} \times \frac{p(D|M_i,I)}{p(D|M_j,I)}
\]

The data-dependent part is called the **Bayes factor**:

\[
B_{ij} \equiv \frac{p(D|M_i,I)}{p(D|M_j,I)}
\]

It is a *likelihood ratio*; the BF terminology is usually reserved for cases when the likelihoods are marginal/average likelihoods.
An Automatic Occam’s Razor

Predictive probabilities can favor simpler models

\[ p(D|M_i) = \int d\theta_i \, p(\theta_i|M) \, \mathcal{L}(\theta_i) \]
The Occam Factor

Models with more parameters often make the data more probable — *for the best fit*

Occam factor penalizes models for “wasted” *volume of parameter space*

Quantifies intuition that models shouldn’t require fine-tuning
Model Averaging

**Problem statement**

\[ I = (M_1 \lor M_2 \lor \ldots) \] — Specify a set of models

Models all share a set of “interesting” parameters, \( \phi \)

Each has different set of nuisance parameters \( \eta_i \) (or different prior info about them)

\( H_i = \) statements about \( \phi \)

**Model averaging**

Calculate posterior PDF for \( \phi \):

\[
p(\phi|D, I) = \sum_i p(M_i|D, I) p(\phi|D, M_i)
\]

\[
\propto \sum_i \mathcal{L}(M_i) \int d\eta_i p(\phi, \eta_i|D, M_i)
\]

The model choice is a (discrete) nuisance parameter here.
Theme: Parameter Space Volume

Bayesian calculations sum/integrate over parameter/hypothesis space!

(Frequentist calculations average over sample space & typically optimize over parameter space.)

- Credible regions integrate over parameter space.
- Marginalization weights the profile likelihood by a volume factor for the nuisance parameters.
- Model likelihoods have Occam factors resulting from parameter space volume factors.

Many virtues of Bayesian methods can be attributed to this accounting for the “size” of parameter space. This idea does not arise naturally in frequentist statistics (but it can be added “by hand”).
Roles of the prior

Prior has two roles

- Incorporate any relevant prior information
- Convert likelihood from “intensity” to “measure” → account for size of parameter space

Physical analogy

\[
Q = \int d\mathbf{r} \ c_v(\mathbf{r}) \ T(\mathbf{r})
\]

\[
P \propto \int d\theta \ p(\theta) \ L(\theta)
\]

Maximum likelihood focuses on the “hottest” parameters. Bayes focuses on the parameters with the most “heat.”

A high-\(T\) region may contain little heat if its \(c_v\) is low or if its volume is small.

A high-\(L\) region may contain little probability if its prior is low or if its volume is small.
Recap of Key Ideas

Probability as generalized logic
Probability quantifies the *strength of arguments*
To appraise hypotheses, calculate probabilities for arguments from data and modeling assumptions to each hypothesis
Use *all* of probability theory for this

Bayes’s theorem

\[ p(\text{Hypothesis} \mid \text{Data}) \propto p(\text{Hypothesis}) \times p(\text{Data} \mid \text{Hypothesis}) \]

Data change the support for a hypothesis \( \propto \) ability of hypothesis to predict the data

Law of total probability

\[ p(\text{Hypotheses} \mid \text{Data}) = \sum p(\text{Hypothesis} \mid \text{Data}) \]

The support for a *compound/composite* hypothesis must account for all the ways it could be true