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ASYMPTOTIC REPRESENTATIONS RELATED TO JACKKNIFING AND BOOTSTRAPPING L-STATISTICS

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SUMMARY. Some asymptotic representations are obtained for the jackknife and the bootstrap procedures on linear combination of order statistics. The results imply asymptotic validity of the jackknife estimate of standard error and the bootstrap approximation to the actual distribution of the statistics.

1. Introduction and the results

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population F with finite second moment. Let $G(t) = \inf\{x : F(x) \geqslant t\}$, $0 < t < 1$, denote the left continuous inverse of F. If U is a random variable having the distribution $U[0, 1]$, then $P(G(U) \leqslant x)) = F(x)$ for all x; hence we can assume without loss of generality that $X_i = G(U_i)$ where $U_1, U_2, \ldots$ are i.i.d. r.v.'s with the distribution $U[0, 1]$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ and $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ denote the order statistics of the samples $X_1, X_2, \ldots, X_n$ and $U_1, U_2, \ldots, U_n$ respectively. It is well known that the estimators based on the jackknife procedure do not behave properly in the case of a single order statistic. An easy way to verify this fact is through the representation

$$\{U_{(1)}, U_{(2)}, \ldots, U_{(n)}\} \overset{\mathcal{D}}{=} \{S_1/S_{n+1}, S_2/S_{n+1}, \ldots, S_n/S_{n+1}\}$$

where $S_1, S_2, \ldots$ is the sequence of partial sums of i.i.d. r.v.'s from the exponential distribution. In Section (5.2) of the survey paper, Miller (1974) wonders whether the jackknife procedure provides valid estimates if one has a smooth linear combination of order statistics. Let us consider the following two L-statistics

$$L_n(w) = \sum_{i=1}^{n} X_{(i)} \int_{(i-1)/n}^{i/n} w(t)dt \qquad \ldots \quad (1)$$

and

$$T_n(w) = \frac{1}{n} \sum_{i=1}^{n} X_{(i)}\, w(i/n) \qquad \ldots \quad (2)$$

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where w is a bounded function on $(0, 1)$ and it is continuous throughout except possibly at finitely many points $a_1 < a_2 < \dots < a_k$ where it is assumed to be either left continuous or right continuous.

Following the standard jackknife procedure, we define $L_{n,-i}(w)$ and $T_{n,-i}(w)$ as in (1) and (2) using $(n-1)$ observations remaining after deleting X_i. Also we define the so called pseudo values

$$L_n^{(i)}(w) = nL_n(w)-(n-1)L_{n,-i}(w)$$

and

$$T_n^{(i)}(w) = nT_n(w)-(n-1)T_{n,-i}(w).$$

Let $Z(X, F, w) = \int_{-\infty}^{\infty} [F(x)-I(X \leqslant x)]w(F(x))dx$ and $L(w) = \int_0^1 G(u)w(u)du$.

Thornburn (1976), Cheng (1982) and Parr and Schucany (1982) have studied the performance of the jackknife technique on L-statistics. Cheng (1982) established the (first-order) asymptotic equivalence of $n^{-1} \sum_1^n L_n^{(i)}$ to L_n under smoothness conditions on the weight function. In Singh (1981a) we have similar results in the case of T_n. Thornburn (1976) proved similar equivalence result under much stronger conditions. Parr and Schucany (1982) also studied the problem of consistency of the jackknife estimator of (asymptotic) variance. They noted that if T_n is the version of L-statistics being jackknifed then in some cases the jackknife estimator of the variance is consistent though $n^{-1} \sum_1^n T_n^{(i)}$ does not retain the asymptotic characters of T_n. No general explanation was offered in this paper for this somewhat curious phenomenon. We present below two representation theorems for the pseudo values which imply strong consistency of the jackknife estimator of variance and Theorem 2 explains why in some cases $n^{-1} \sum_1^n T_n^{(i)}$ is not equivalent to T_n. Also, in many respects the conditions here are weaker than those in Parr and Schucany (1982). In particular we do not require L-statistics to be of trimmed type as required in Theorem 2 of Parr and Schucany (1982). See also Note 4 of Theorem 2 of Parr and Schucany (1982).

Theorem 1 : *If w satisfies a Lipschitz condition of order* 1 *on each of the intervals (a_{i-1}, a_i), $i = 1, 2, \dots, k+1$ ($a_0 = 0$ and $a_{k+1} = 1$), and G is continuous at $a_1, a_2, \dots, a_k$, then*

$$\max_{1 \leqslant i \leqslant n} |L_n^{(i)}(w)-L(w)-Z(X_i, F, w)| \to 0 \text{ a.s.}$$

Theorem 2 : *If w has a bounded continuous derivative on each of the intervals (a_{i-1}, a_i), $i = 1, 2, \ldots, k+1$ and G is continuous at $a_1, a_2, \ldots, a_k$, then*

$$\max_{1 \leq i \leq n} |T_n^{(i)}(w) - L(w) - Z(X_i, F, w) - \xi(F, w, n)| \to 0 \text{ a.s.}$$

where $\xi(F, w, n)$ is a non-random quantity depending only upon F, w and n.

To be more precise $\xi(F, w, n)$ is a function of a_i, $[w(a_i+) - w(a_i-)]$ and $G(a_i)$, $i = 1, 2, \ldots, k$. It is defined in the proof. Indeed, $\xi(F, w, n) = 0$ if w has no jump. Usually, $\xi(F, w, n)$ is a periodic function of n if the jump points of w are rationals.

Under the above set up (see (P9)),

$$L_n(w) - L(w) = \bar{Z}_n + o_p(n^{-\frac{1}{2}}); \; T_n(w) - L(w) = \bar{Z}_n + o_p(n^{-\frac{1}{2}})$$

where $\bar{Z}_n = \frac{1}{n} \sum_1^n Z(X_i, F, w)$. Also $Z(X_1, F, w)$ has mean zero and finite variance (see (P7)). Consequently, Theorem 1 implies that if $Z(X, F, w)$ is non-degenerate (we assume this hereafter), then

$$\sqrt{n}[L_n(w) - L(w)]/l_n \xrightarrow{\mathcal{D}} N(0, 1)$$

where
$$l_n^2 = \frac{1}{n} \sum_1^n \left[L_n^{(i)}(w) - \frac{1}{n} \sum_1^n L_n^{(i)}(w) \right]^2.$$

Since the non-random quantity $\xi(F, w, n)$ is free from i, similar conclusions regarding $T_n(w)$ follow from Theorem 2. We thus see that the jackknife is sensitive to the variations in the definition of the statistics.

We now discuss the bootstrap procedure on $L_n(w)$ and $T_n(w)$. Let F_n denote the empirical d.f. based on $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ denote a random sample of size n from F_n. Then, by definition (see Efron, 1979), the bootstrap distribution of $L_n(w) - L(w)$ is the distribution of $L_n^*(w) - L_n(w)$ under F_n where

$$L_n^*(w) = \sum_{i=1}^n Y_{(i)} \int_{(i-1)/n}^{i/n} w(t)dt,$$

and $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are the order statistics of $Y_1, \ldots, Y_n$. We establish the following representation for $L_n^*(w)$.

Theorem 3 : *Under the conditions of Theorem* 1,

$$L_n^*(w) - L_n(w) = \frac{1}{n} \sum_{i=1}^n Z(Y_i, F, w) - \frac{1}{n} \sum_{i=1}^n Z(X_i, F, w) + o_p^*(n^{-\frac{1}{2}})$$

where o_p^* refers to probability under F_n.

This theorem together with bootstrap CLT (see Theorem 1.1 of Singh, 1981b; or Theorem 2.1 of Bickel and Freedman, 1981) yields the following

$$\sup_x |P^*(L_n^*(w)-L_n(w) \leqslant x)-P(L_n(w)-L(w) \leqslant x)| \to 0 \text{ a.s.} \qquad \ldots (3)$$

where P^* refers to probability under F_n. Also, if we define

$$T_n^*(w) = \frac{1}{n} \sum_{i=1}^{n} Y_{(i)} w(i/n),$$

then we have

Theorem 4 : *Under the conditions of Theorem* 1, *a.s.*

$$T_n^*(w)-T_n(w) = n^{-1} \sum_{i=1}^{n} Z(Y_i, F, w)-n^{-1} \sum_{i=1}^{n} Z(X_i, F, w)+o_p^*(n^{-\frac{1}{2}}).$$

Of course, Theorem 4 also yields a conclusion similar to (3) for $T_n(w)$.

Incidentally, an inspection of the proofs in the following sections show that the condition $EX_1^2 < \infty$ is superfluous if w trims the data.

Finally, we consider a class of L-statistics of a different sort. Let μ be a function of bounded variation defined over $[\alpha, \beta]$, $0 < \alpha < \beta < 1$ and let F_n^* denote e.d.f. based on a random sample of size n from F_n. Also define

$$F_n^{-1}(t) = \inf\{x : F_n(x) \geqslant t\},$$

$$F_n^{*-1}(t) = \inf\{x : F_n^*(x) \geqslant t\}, \quad 0 < t < 1,$$

$$M_n(\mu) = \int_\alpha^\beta F_n^{-1}(t) d\mu(t),$$

$$M(\mu) = \int_\alpha^\beta G(t) d\mu(t),$$

$$f_t = F'(G(t)) \text{ (when it exists)}$$

and

$$\eta(X, F, \mu) = \int_\alpha^\beta ([t-I(X \leqslant G(t))]/f_t) d\mu(t).$$

A bootstrap version of Bahadur-Kiefer representation of quantiles is stated below, which implies the bootstrap CLT for $M_n(\mu)$. The bootstrap CLT for $M_n(\mu)$ also follows from the functional limit theorem given as Theorem 5.1 in Bickel and Freedman (1981). The next theorem does not require any moment condition on X_1.

Theorem 5 : *Assume that, for some* $\varepsilon > 0$, *on* $[G(\alpha-\varepsilon), G(\beta+\varepsilon)]$, F *is twice differentiable* F'' *is bounded and* F' *is bounded away from zero. Then a.s.*,

$$\sup_{\alpha \leqslant t \leqslant \beta} |F_n^{*-1}(t)-F_n^{-1}(t)+([F_n^*(G(t))-F_n(G(t))]/f_t)| = O_p^*(n^{-3/4} \log n).$$

As a consequence, a.s.

$$M_n^*(\mu)-M_n(\mu) = \frac{1}{n}\sum_{i=1}^{n}\eta(Y_i, F, \mu)-\frac{1}{n}\sum_{i=1}^{n}\eta(X_i, F, \mu)+O_p^*(n^{-3/4}\log n).$$

The rate of convergence in Theorem 5 as stated is slightly worse (at log term) than the usual rates in Bahadur-Kiefer representations. Possibly the same rates can be achieved by Theorem 5 but we do not pursue the question here.

2. Preliminaries

In this section we collect some results, which are required for the proofs of the theorems. These are stated as (P1) to (P10) with a sketch of proof in most cases.

(P1). Let H_1^{-1} and H_2^{-1} be the left continuous inverse functions of two d.f.'s H_1 and H_2 with finite first moments. Let θ be a bounded measurable function defined on (0, 1), then

$$\int_0^1 (H_1^{-1}(t)-H_2^{-1}(t))\theta(t)dt = -\int_{-\infty}^{\infty}[\lambda(H_1(x))-\lambda(H_2(x))]\,dx,$$

where $\lambda(t) = \int_0^t \theta(y)dy$. The identity follows from integration by parts. In the context of L-statistics this equality has been used in Boos (1979).

(P2). If $EX_1^2 < \infty$, then $n^{-\frac{1}{2}}(|X_{(1)}|+|X_{(n)}|)\to 0$ a.s. This follows from the fact that

$$\sum_{i=1}^{\infty} P(|X_i| > \varepsilon i^{\frac{1}{2}}) < \infty \text{ for all } \varepsilon > 0.$$

(P3). $\sup_x |F_n(x)-F(x)| \leqslant n^{-\frac{1}{2}}\log n$, for all large n, a.s.

(P4). If $EX_1^2 < \infty$, then $\int_{-\infty}^{\infty} |F_n(x)-F(x)|\,dx\to 0$ a.s. This can be deduced using (P2),

$$\int_{-\infty}^{\infty} F(x)(1-F(x))dx < \infty$$

and

$$\sup_{-\infty<x<\infty} |F_n(x)-F(x)|\,[F(x)(1-F(x))]^{-\frac{1}{2}} = O(n^{-\frac{1}{2}}\log n) \text{ a.s.}$$

(see Theorem 3.1 (11) of Csáki, 1975).

(P5). For any $a \in (0, 1)$,

$$\lim_{\varepsilon\to 0}\ \limsup_{n\to\infty}\ \frac{1}{n}\sum_{(i/n)\in[a-\varepsilon,\, a+\varepsilon]} |X_{(i)}| = 0 \text{ a.s.}$$

We can show this as follows :

Since $$\max_{1\leqslant i\leqslant n} |U_{(i)}-i/n| \to 0 \text{ a.s.}, \qquad \dots (4)$$

(see Lemma 3.2 of Singh, 1981a for a reference),

$$\frac{1}{n}\sum_{(i/n)\in[a-\varepsilon,\, a+\varepsilon]} |X_{(i)}| \leqslant \frac{1}{n}\sum_{i=1}^{n} |G(U_i)|\, I(U_i \in [a-2\varepsilon,\, a+2\varepsilon]).$$

Now the claim follows from SLLN.

(P6). If G is continuous at a, $0 < a < 1$, then

$$\max_{|l|\leqslant k} |X_{[na]+l}-G(a)| \to 0 \quad \text{a.s.}$$

for any positive integer k. To see this we once again appeal to (4) and conclude that for a given $\varepsilon > 0$,

$$\max_{|l|\leqslant k} |X_{[na]+l}-G(a)| \leqslant \max_{|l|\leqslant k} |G(U_{([na]+l)})-G(a)|$$

$$\leqslant |G(a+\varepsilon)-G(a-\varepsilon)| \text{ for all large } n \text{ a.s.}$$

(P7). $\int_{-\infty}^{\infty} |F(x)-I(X_i \leqslant x)|\, dx \leqslant 2(|X_i|+E|X_i|)$. To prove this claim we argue, using integration by parts, as follows. The LHS of (P7) is

$$\leqslant \int_{-\infty}^{X_i} F(x)dx + \int_{X_i}^{\infty} (1-F(x))\, dx$$

$$\leqslant X_iF(X_i) - \int_{-\infty}^{X_i} x\, dF(x) - X_i(1-F(X_i)) + \int_{X_i}^{\infty} x\, dF(x)$$

$$\leqslant 2[|X_i|+E|X_i|].$$

(P8). $$\sqrt{n} \sup_{-\infty < x < \infty} |F_n^*(x)-F_n(x)| = O_p^*(1) \text{ a.s.}$$

This is an immediate consequence of Theorem 4.1 of Bickel and Freedman (1981).

(P9). Under the conditions of Theorem 1,

$$L_n(w)-L(w) = \frac{1}{n}\sum_{1}^{n} Z(X_i, F, w)+o_p(n^{-\frac{1}{2}})$$

and $$T_n(w)-L(w) = \frac{1}{n}\sum_{1}^{n} Z(X_i, F, w)+o_p(n^{-\frac{1}{2}}).$$

This kind of representations have been studied in literature under various conditions (see Singh, 1981a). A proof of (P9) is given after the proof of Theorem 4 in Section 3.

(P10). If $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ denote the order statistics of the bootstrap sample $(Y_1, Y_2, \ldots, Y_n)$ from F_n then, for any $a \in (0, 1)$ and a positive integral k,

$$\max_{|l| \leqslant k} |Y_{[na]+l}| = O_p^*(1) \text{ a.s.}$$

To see this note that

$$P^*(Y_{[na]+k} < G(a-\varepsilon)-\varepsilon) \leqslant P^*(F_n^*(G(a-\varepsilon)-\varepsilon) \geqslant (na+l)/n)$$

$$\leqslant P^*(F_n^*(G(a-\varepsilon)-\varepsilon)-F_n(G(a-\varepsilon)-\varepsilon) \geqslant \varepsilon/2)$$

for all large n, a.s. Thus $P^*(Y_{[na]+k} < G(a-\varepsilon)-\varepsilon) \to 0$ a.s., by (P8). Similarly $P^*(Y_{[na]-k} > G(a+\varepsilon)+\varepsilon) \to 0$ a.s. Hence the claim.

3. Proofs of the theorems

Proof of Theorem 1 : (P1) together with the definition of $L_n^{(i)}(w)$ gives

$$L_n^{(i)}(w)-L(w) = [L_n(w)-L(w)]+(n-1)[L_n(w)-L(w)]-(n-1)[L_{n,-i}(w)-L(w)]$$

$$= L_n(w)-L(w)-(n-1)\int_{-\infty}^{\infty}[W(F_n(x))-W(F_{n,-i}(x))]dx$$

where $W(t) = \int_0^t w(y)dy$, $0 \leqslant t \leqslant 1$, and $F_{n,-i}$ is the e.d.f. based on the $(n-1)$ observations $\{X_1, X_2, \ldots, X_n\}-\{X_i\}$. Thus, in view of (P1) and (P4), the theorem would follow if we show

$$\max_{1\leqslant i\leqslant n}\left| Z(X_i, F, w)+\int_{-\infty}^{\infty}(n-1)[W(F_n(x))-W(F_{n,-i}(x))]dx\right| \to 0 \text{ a.s.} \quad \ldots \quad (5)$$

Let

$$C_n = \bigcup_{j=1}^{k}[G(a_j-2\beta_n)-\beta_n, G(a_j+2\beta_n)+\beta_n]$$

where $\beta_n = n^{-\frac{1}{2}}\log n$. Now, W satisfies a Lipschitz condition of order 1 throughout, $|F_n(x)-F_{n,-i}(x)| \leqslant (n-1)^{-1}$ for all x and G is continuous at $a_1, a_2, \ldots, a_k$. These facts tell us that

$$\max_{1\leqslant i\leqslant n}|(n-1)\int_{C_n}[W(F_n(x))-W(F_{n,-i}(x))]\, dx| \to 0 \text{ a.s.}$$

Thus to arrive at (5) we only need to see that

$$\max_{1\leqslant i\leqslant n}| Z(X_i, F, w)+\int_{(-\infty,\infty)-C_n}(n-1)[W(F_n(x))-W(F_{n,-i}(x))]dx| \to 0 \text{ a.s.}$$

Towards this end, let us observe, using (P3) that if $x \notin C_n$ then $F_n(x)$, $F_{n,-i}(x)$ and $F(x)$ all three lie in an interval (a_i, a_{i+1}) for some $0 \leqslant i \leqslant k$, for all $n \geqslant n_0$

(independent of x). This along with the mean value theorem and (P7) enables us to write

$$\int_{(-\infty,\,\infty)-C_n} [W(F_n(x))-W(F_{n,-i}(x))]dx = \int_{(-\infty,\,\infty)-C_n} (F_n(x)-F_{n,-i}(x))w(F_n(x))dx+\gamma_n$$

where $\gamma_n = O\left(\frac{1}{n}\right)\int_{-\infty}^{\infty} |F_n(x)-F_{n,-i}(x)|\,dx = O\left(n^{-2}\left(\max_{1\leqslant j\leqslant n} |X_j|+E|X_1|\right)\right)$

Thus by (P2) $\gamma_n = O(n^{-3/2})$ a.s. Now (P4) gives, uniformly in $1 \leqslant i \leqslant n$, a.s.

$$-(n-1)\int_{(-\infty,\,\infty)-C_n} (F_n(x)-F_{n,-i}(x))w(F_n(x))dx$$

$$= -(n-1)\int_{(-\infty,\,\infty)-C_n} (F_n(x)-F_{n,-i}(x))w(F(x))dx+o(1)$$

$$= -(n-1)\int_{-\infty}^{\infty} [F_n(x)-F(x)+F(x)-F_{n,-i}(x)]w(F(x))dx+o(1)$$

$$= -n\int_{-\infty}^{\infty} [F_n(x)-F(x)]w(F(x))dx+(n-1)\int_{-\infty}^{\infty} [F_{n,-i}(x)-F(x)]w(F(x)dx+o(1)$$

$$= Z(X_i, F, w)+o(1).$$

This completes the proof.

Proof of Theorem 2: Adopting the notation $J_l = [w(a_l+)-w(a_l-)]$, $1 \leqslant l \leqslant k$, let us note that we can decompose w as $w_0+\sum_{l=1}^{k} w_l$ where w_0 is a continuous function on $(0, 1)$ with bounded continuous derivative over $\bigcup_{1}^{k+1} (a_{l-1}, a_l)$; $w_l = J_l I(a_l, 1)$ if w is left continuous at a_l, $= J_l I[a_l, 1)$ if w is right continuous at a_l. We deduce Theorem 2 from Theorem 1 and the following:

$$\max_{1\leqslant i\leqslant n} |L_n^{(i)}(w_0)-T_n^{(i)}(w_0)| \to 0 \text{ a.s.} \qquad \ldots \quad (6)$$

and

$$\max_{1\leqslant i\leqslant n} |L_n^{(i)}(w_l)-T_n^{(i)}(w_l)-\xi_l(F, w, n)| \to \text{ a.s.} \qquad \ldots \quad (7)$$

for all $1 \leqslant l \leqslant k$, where $\xi_l(F, w, n)$ is a non-random function of F, w and n.

Let us look at (6) first. For $0 < \varepsilon < \min\{|a_i-a_{i-1}|;\ i = 1, 2, \ldots, k+1\}$, define $E_\varepsilon = \bigcup_{i=0}^{k+1} [a_i-\varepsilon, a_i+\varepsilon]$. If $\frac{i}{n} \notin E_\varepsilon$, then for all large n, $\frac{i}{n}, \frac{i-1}{n}, \frac{i}{n-1}, \frac{i+1}{n-1}, \frac{i+2}{n-1}$ all belong to $(0, 1)-E_{\varepsilon/2}$. Note that w_0 satisfies Lipschitz condition of order 1 on $(0, 1)$. Let

$$b_j = n\int_{(j-1)/n}^{j/n} (w_0(t)-w_0(j/n))dt-(n-1)\int_{(j-1)/(n-1)}^{j/(n-1)} (w_0(t)-w_0(j/(n-1)))dt$$

and

$$c_j = n\int_{(j-1)/n}^{j/n} (w_0(t)-w_0(j/n))dt-(n-1)\int_{(j-2)/(n-1)}^{(j-1)/(n-1)} (w_0(t)-w_0((j-1)/(n-1)))dt.$$

By Taylor's expansion there exists a constant $c > 0$ such that

$$|b_j|+|c_j| \leqslant \begin{cases} c/n, & \text{if } (j/n) \in E_\varepsilon, \\ cg_n(\varepsilon)/n, & \text{otherwise} \end{cases} \qquad \dots \quad (8)$$

where, for every $\varepsilon > 0$,

$$g_n(\varepsilon) = \sup\{|w_0^1(t)-w_0^1(s)| : t, s \in (0,1)-E_{\varepsilon/2},\ |t-s| < 2/n\} \to 0 \qquad \dots \quad (9)$$

as $n \to \infty$. Also, let $d_i = \#\{j : X_j = X_i, 1 \leqslant j \leqslant n\}$ and $k+1, k+2, \dots, k+d_i$ be the ranks assigned to these X_j. Now by (8),

$$|L_n^{(i)}(w_0)-T_n^{(i)}(w_0)| \leqslant |\sum_{X_{(j)}<X_i} b_j X_{(j)}$$

$$+\sum_{X_{(j)}>X_i} c_j X_{(j)} + X_i \sum_{j=k+1}^{(k+d_i)-1} b_j + nX_i \int_{(k+d_i-1)/n}^{(k+d_i)/n} (w(t)-w((k+d_i)/n))dt|$$

$$\leqslant \frac{\text{const}}{n}\Big[g_n(\varepsilon)\sum_{(j/n)\in(0,1)-E_\varepsilon} |X_{(j)}| + \sum_{(j/n)\in E_\varepsilon} |X_{(j)}| + |X_{(1)}| + |X_{(n)}|\Big].$$

The claim (6) follows now from (P2) and (P5).

Turning to (7), note that for $l \geqslant 1$,

$$\int_{(i-1)/n}^{i/n} (w_l(t)-w_l(i/n))dt = \begin{cases} 0 & \text{if } i \geqslant na_l+1 \text{ or } < na_l, \\ -J_l(a_l-[na_l]/n) & \text{if } i = [na_l]+1, \\ 0 & \text{if } i = na_l \text{ and } w \text{ is left continuous at } a_l, \\ -J_l/n & \text{if } i = na_l \text{ and } w \text{ is right continuous at } a_l. \end{cases} \qquad \dots \quad (10)$$

Let

$$f(J_l, a_l, n) = \begin{cases} -J_l(na_l-[na_l]) & \text{if } w \text{ is left continuous at } a_l, \\ -J_l\{(na_l-[na_l])+1\} & \text{if } w \text{ is right continuous at } a_l. \end{cases}$$

These facts together with (P6) tell us that a.s.,

$$|n[L_n(w_l)-T_n(w_l)]-G(a_l)f(J_l, a_l, n)| \to 0. \qquad \dots \quad (11)$$

By the same token a.s.,

$$\max_{1\leqslant i\leqslant n} |(n-1)[L_{n,-i}(w_l)-T_{n,-i}(w_l)]-G(a_l)f(J_l, a_l, n-1)| \to 0. \qquad \dots \quad (12)$$

Clearly (11) and (12) yield (7). The proof of Theorem 2 is completed by taking

$$\zeta(F, w, n) = \sum_{l=1}^{k} (Ga_l)[f(J_l, a_l, n) - f(J_l, a_l, n-1)].$$

It may be mentioned here that Theorem 2 can be proved alternatively, under possibly slightly stronger conditions, using the representation techniques of Singh (1981a).

Proof of Theorem 3 : Since

$$n^{-1} \sum_{i=1}^{n} [Z(Y_i, F, w) - Z(X_i, F, w)] = -\int_{-\infty}^{\infty} [F_n(x) - F_n^*(x)] w(F(x)) dx,$$

we have to show that a.s.

$$R_n(w) = o_p^*(n^{-\frac{1}{2}})$$

where $$R_n(w) = \left| L_n^*(w) - L_n(w) + \int_{-\infty}^{\infty} [F_n^*(x) - F_n(x)] w(F(x)) dx \right|. \qquad \ldots \quad (13)$$

Let us recall the decomposition $w = w_0 + \sum_{l=1}^{k} w_l$ mentioned in the proof of Theorem 2. We show (13) replacing w by w_l, for $0 \leqslant l \leqslant k$, separately.

Let us define $W_l(t) = \int_0^t w_l(y) dy$; $l = 0, 1, 2, \ldots, k$. Using (P1), (P2), (P4), (P8) and the fact that w_0 satisfies a Lipschitz condition of order 1 on (0, 1) we get the following estimates of $R_n(w_0)$.

$$\begin{aligned} R_n(w_0) &= \left| \int_{-\infty}^{\infty} [W_0(F_n(x)) - W_0(F_n^*(x)) - (F_n(x) - F_n^*(x)) w_0(F(x))] dx \right| \\ &\leqslant \text{const} \Big[\int_{-\infty}^{\infty} |F_n(x) - F_n^*(x)|^2 dx + \int_{-\infty}^{\infty} |F_n(x) - F_n^*(x)| \, |F_n(x) - F(x)| \, dx \\ &\leqslant \text{const} \Big[(\sup_x |F_n(x) - F_n^*(x)|^2)(|X_{(1)}| + |X_{(n)}|) \\ &\qquad + \sup_x |F_n(x) - F_n^*(x)| \Big(\int_{-\infty}^{\infty} |F_n(y) - F(y)| dy \Big) \Big] \\ &= o_p^*(n^{-\frac{1}{2}}) \text{ a.s.} \end{aligned}$$

Let $$I_1 = (-\infty, G(a_l - 3\beta_n) - \beta_n], \; I_3 = [G(a_l + 3\beta_n) + \beta_n, \infty)$$

and $$I_2 = (-\infty, \infty) - (I_1 \cup I_3).$$

Notice that for $l \geqslant 1$,

$$\begin{aligned} R_n(w_l) &\leqslant \Big(\int_{I_1} + \int_{I_2} + \int_{I_3} \Big) | W_l(F_n(x)) - W_l(F_n^*(x)) - (F_n(x) - F_n^*(x)) w_l(F(x)) | \, dx \\ &= I + II + III \text{ (say).} \end{aligned}$$

That $II = o_p^*(n^{-\frac{1}{2}})$ is immediate from (P8) and the facts that G is continuous at a_l and W_l satisfies Lipschitz condition of order 1. If $\sup_x |F_n(x)-F(x)| \leqslant \beta_n$, which is true for all large n a.s., and $\sup_x |F_n(x)-F_n^*(x)| \leqslant \beta_n$, then it is easy to see that $I+III = 0$. Therefore

$$P^*(I+III \geqslant n^{-\frac{1}{2}}\varepsilon) \leqslant P^*(\sup_x |F_n(x)-F_n^*(x)| \geqslant \beta_n) \to 0$$

by (P8). Hence the theorem.

Proof of Theorem 4 : Theorem 3 and the following yield Theorem 4.

$$n|T_n(v)-L_n(v)| = O(1) \text{ a.s.} \quad \ldots \quad (14)$$

and

$$n|T_n^*(v)-L_n^*(v)| = O_p^*(1) \text{ a.s.} \quad \ldots \quad (15)$$

with $v = w_0, w_1, \ldots, w_l$. If $v = w_l$, $l \geqslant 1$, then (14) and (15) follow from (P6), P(10) and (10). In the case of $v = w_0$, (14) and (15) follow, as w_0 satisfies Lipschitz condition of order 1. This completes the proof.

Proof of P(9) : We shall be using the notations developed in the earlier proofs of this section. First note that the second part of P(9) follows from the first part and (14). To deduce the first part we show the following : For $v = w_0, w_1, \ldots, w_k$, a.s.,

$$r_n(v) = |L_n(v)-L(v)-n^{-1}\sum_1^n Z(X_i, F, v)| = o_p(n^{-\frac{1}{2}}).$$

In the case $v = w_0$, we have, in view of P(1) and Csáki's result used in P(4),

$$r_n(w_0) = |\int_{-\infty}^{\infty} [W_0(F_n(x))-W_0(F(x))-(F_n(x)-F(x))w_0(F(x))]dx|$$

$$\leqslant \text{const.} \int_{-\infty}^{\infty} |F_n(x)-F(x)|^2 dx$$

$$\leqslant o(n^{-\frac{1}{2}}) \int_{-\infty}^{\infty} F(x)[1-F(x)]\,dx = o(n^{-\frac{1}{2}}).$$

For w_l, $l > 0$, we have (recall I_1, I_2, I_3 used in the proof of Theorem 3

$$r_n(w_l) \leqslant \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) |W_l(F_n(x))-W_l(F(x))-(F_n(x)-F(x))w_l(F(x))|\,dx.$$

Now the integral over I_2 is $o_p(n^{-\frac{1}{2}})$ since W_l is Lipschitz of order 1 and $\sqrt{n}\sup_x |F_n(x)-F(x)| = o_p(1)$. The integrals over I_1 and I_3 are in fact zero for all large n, a.s.

We require the following notation to prove the next theorem. Let V_n denote the empirical distribution of $U_1, \ldots, U_n$. Let $Y_i = G(U_i^*)$, $1 \leqslant i \leqslant n$, and V_n^* denote the empirical distribution of $U_1^*, \ldots, U_n^*$.

Proof of Theorem 5 : To prove the theorem we first notice the following identities. $F_n^{*-1}(t) = G(V_n^{*-1}(t))$, $F_n^*(G(t)) = V_n^*(t)$, $F_n^{-1}(t) = G(V_n^{-1}(t))$ and

$F_n(G(t)) = V_n(t)$ over $t \in [\alpha, \beta]$. The last two equalities do require that F is continuous over $[G(\alpha-\varepsilon), G(\beta+\varepsilon)]$ which is implied by the conditions imposod on F. These identities along with the assumed regularity condition on F and

$$\sup_{0<t<1} | V_n^{*-1}(t) - V_n^{-1}(t) + V_n^*(t) - V_n(t) | = O_p^*(n^{-3/4} \log n) \text{ a.s.} \quad \ldots \quad (16)$$

imply the theorem.

To establish (16) we express $V_n^{*-1}(t) - V_n^{-1}(t) + V_n^*(t) - V_n(t)$ as the following sum :

$$\begin{aligned} &\{[V_n^*(t) - V_n(t)] - [V_n^*(V_n^{*-1}(t)) - V_n(V_n^{*-1}(t))]\} \\ &\quad + \{[V_n(t) - t] - [V_n(V_n^{*-1}(t)) - V_n^{*-1}(t)]\} \\ &\quad + \{[V_n^*(V_n^{*-1}(t)) - t]\} + \{-[V_n^{-1}(t) - t] + [t - V_n(t)]\} \\ &= I + II + III + IV \text{ (say).} \end{aligned}$$

The supremum of $|IV|$ over $t \in (0, 1)$ is $O(n^{-3/4} \log n)$ a.s. by a theorem of Kiefer (1970). Following the lines of the arguments in Babu and Singh (1978) it can be shown that suprema of $|I|$, $|II|$, $|III|$ over $t \in (0, 1)$ are $O_p^*(n^{-3/4} \log n)$ a.s. We do not find it appropriate to produce the lengthy details here.

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