Hierarchical Bayesian Modeling with Approximate Bayesian Computation

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Goal of lecture

- Bring the ideas of Hierarchical Bayesian Modeling (HBM), Approximate Bayesian Computation (ABC), importance sampling, and sequential ABC together in an astronomy-motivated application to exoplanets.

Collaborators

Eric Ford

Megan Shabram

Chad Schafer

samsi

NSF • Duke • NCSU • UNC • NISS

SAMSI ExoStat Group

Planet detection

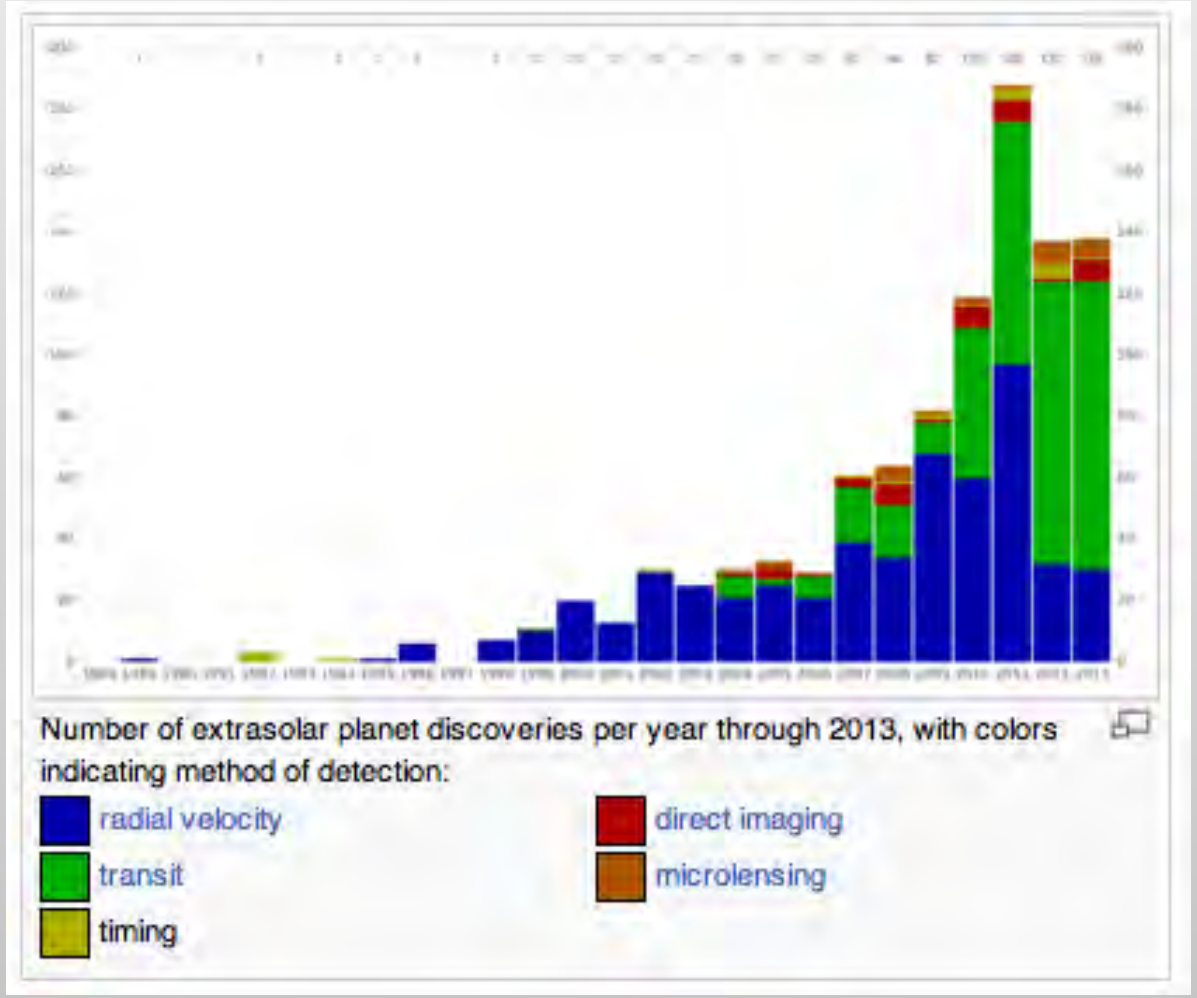
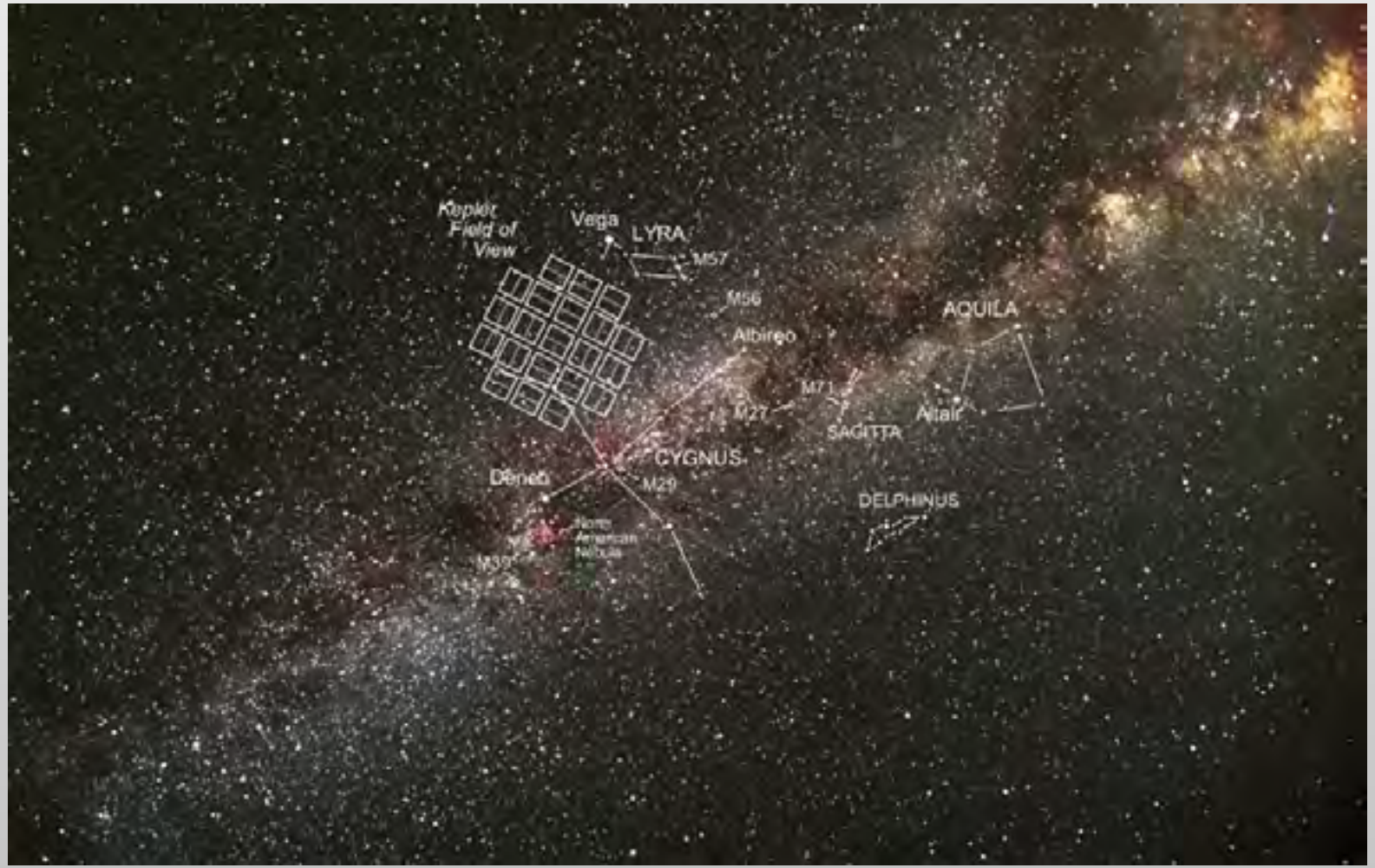


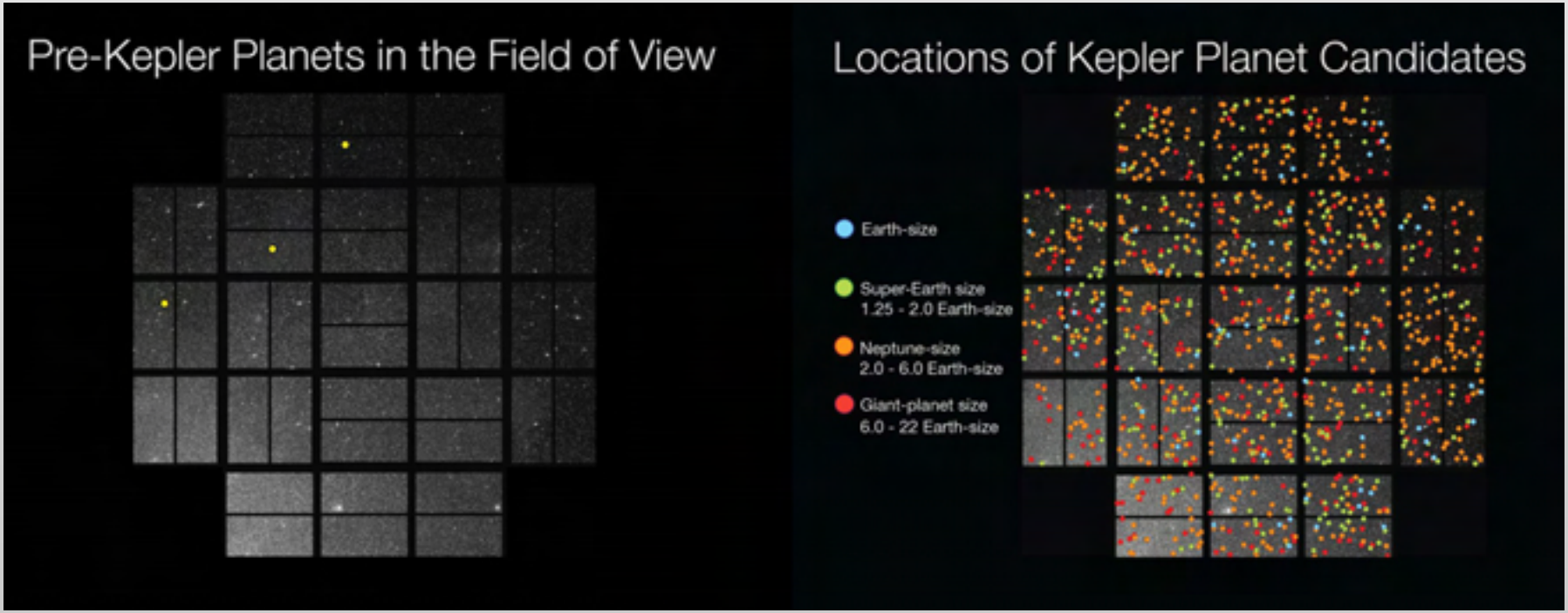
Figure: http://en.wikipedia.org/wiki/File:Exoplanet_Discovery_Methods_Bar.png

Kepler field of view



http://kepler.nasa.gov/images/

Kepler field of view



(pre - 2011)

`http://kepler.nasa.gov/images/`

Each exoplanet, i, has the following parameters:

$$w_i \equiv (R_{s_A}, R_{p_A}, \kappa_i, \Pi_i, \phi_i, e_i, \bar{\omega}_i)$$

- R_{s_i} = star radius
- R_{p_i} = planet radius
- κ_i = amplitude of velocity model
- Π_i = orbital period
- ϕ_i = orbital phase at fiducial time
- e_i = orbital eccentricity
- $\bar{\omega}_i$ = longitude of perihelion (orientation of ellipse relative to the sky)

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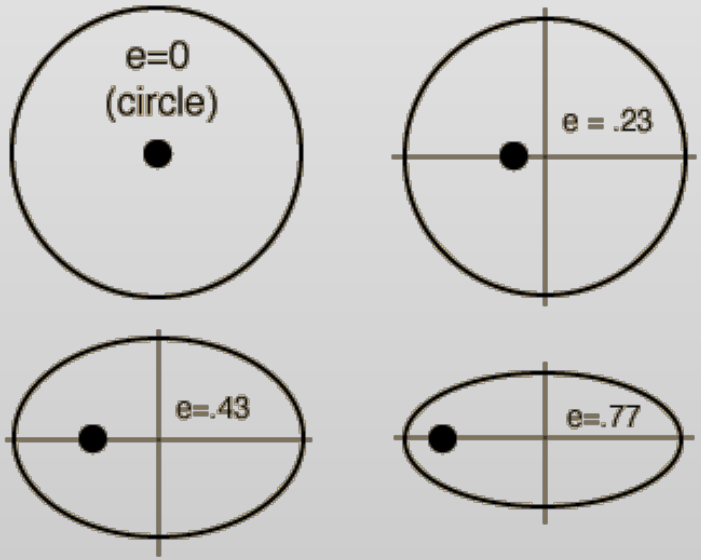
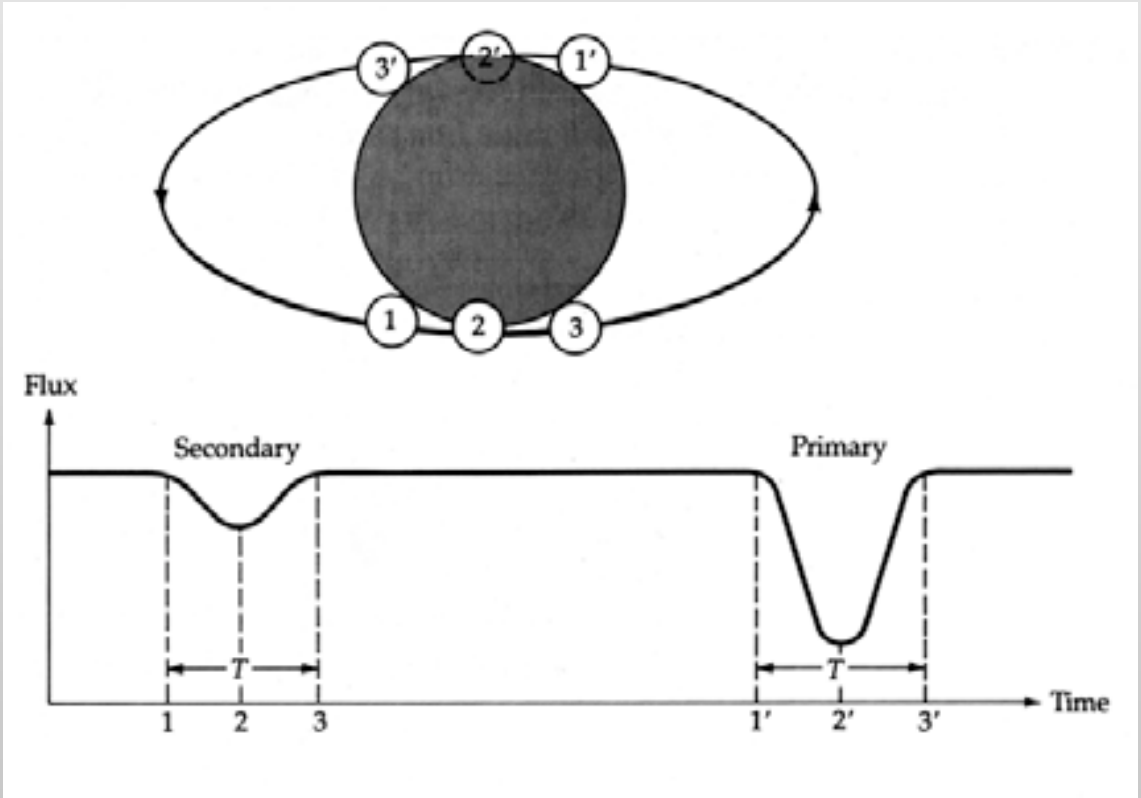
Orbital eccentricity



Planet	Eccentricity
Mercury	0.2056
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0483
Saturn	0.0560
Uranus	0.0461
Neptune	0.0097

Orbital eccentricity



Eccentricity: observables

There is some true (h_i, k_i) for planet i such that

$$\begin{pmatrix} h_i \\ k_i \end{pmatrix} | \text{[some parameters]} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{c_i}^2 & 0 \\ 0 & \sigma_{c_i}^2 \end{pmatrix}\right)$$

- where $h_i = e_i \cos(\bar{\omega}_i)$, $k_i = e_i \sin(\bar{\omega}_i) \implies h_i^2 + k_i^2 = e_i^2$
- $\bar{\omega}_i \sim U[0, 2\pi)$
- But we observe (h_i, k_i) with measurement error

$$\begin{pmatrix} \hat{h}_i \\ \hat{k}_i \end{pmatrix} | \text{[some parameters]} \sim N\left(\begin{pmatrix} h_i \\ k_i \end{pmatrix}, \begin{pmatrix} \sigma_h^2 & 0 \\ 0 & \sigma_k^2 \end{pmatrix}\right)$$

Eccentricity: [some parameters]

Given a sample of exoplanets $p_1, \ldots, p_n$, we assume the following

1. N_m "populations" (assume $N_m = 1, 2,$ or 3)
2. Classify each population as $c_1, \ldots, c_{N_m}$
3. Each population has a certain true proportion f_j, $j = 1, \ldots, N_m$ and $\sum_{j=1}^{N_m} f_j = 1$

Bayesian hierarchical model for eccentricity

$$N_m = \#\text{ "populations" (fixed and "known")}$$

$$f \sim \text{Dirichlet}(1_{N_m})$$

$$c_i \mid f \overset{iid}{\sim} \text{multinomial}(f)$$

$$\sigma_{c_i} \overset{iid}{\sim} U(0,1)$$

$$\begin{pmatrix} h_i \\ k_i \end{pmatrix} | c_i, \sigma_{c_i}^2 \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{c_i}^2 & 0 \\ 0 & \sigma_{c_i}^2 \end{pmatrix}\right)$$

$$\begin{pmatrix} \hat{h}_i \\ \hat{k}_i \end{pmatrix} | c_i, \sigma_{c_i}^2, h_i, k_i \sim N\left(\begin{pmatrix} h_i \\ k_i \end{pmatrix}, \begin{pmatrix} \hat{\sigma}_h^2 & 0 \\ 0 & \hat{\sigma}_k^2 \end{pmatrix}\right)$$

3D Dirichlet Distribution

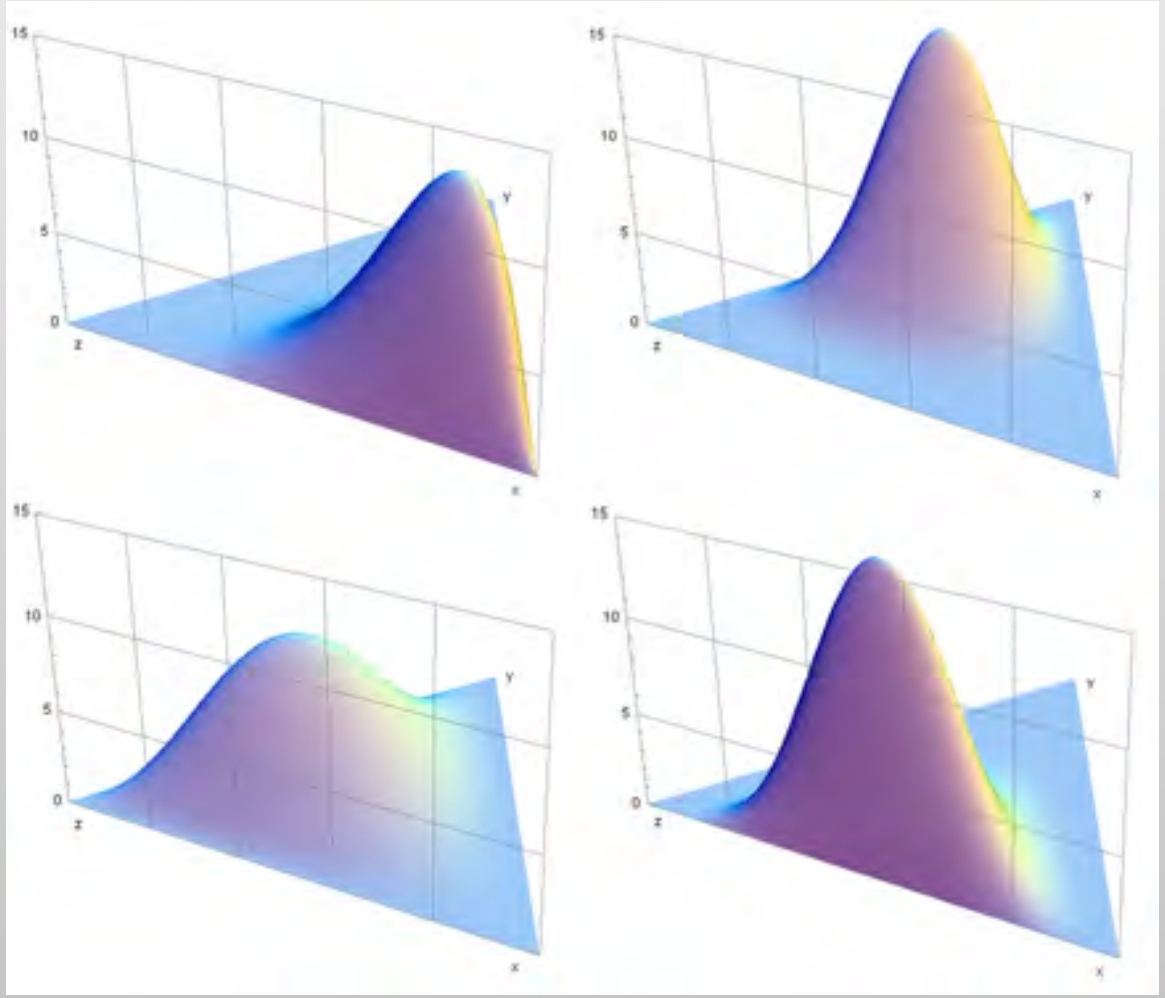


Image: `http://upload.wikimedia.org`

Beta Distribution

$$f(y \mid \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot y^{\alpha-1}(1-y)^{\beta-1}$$

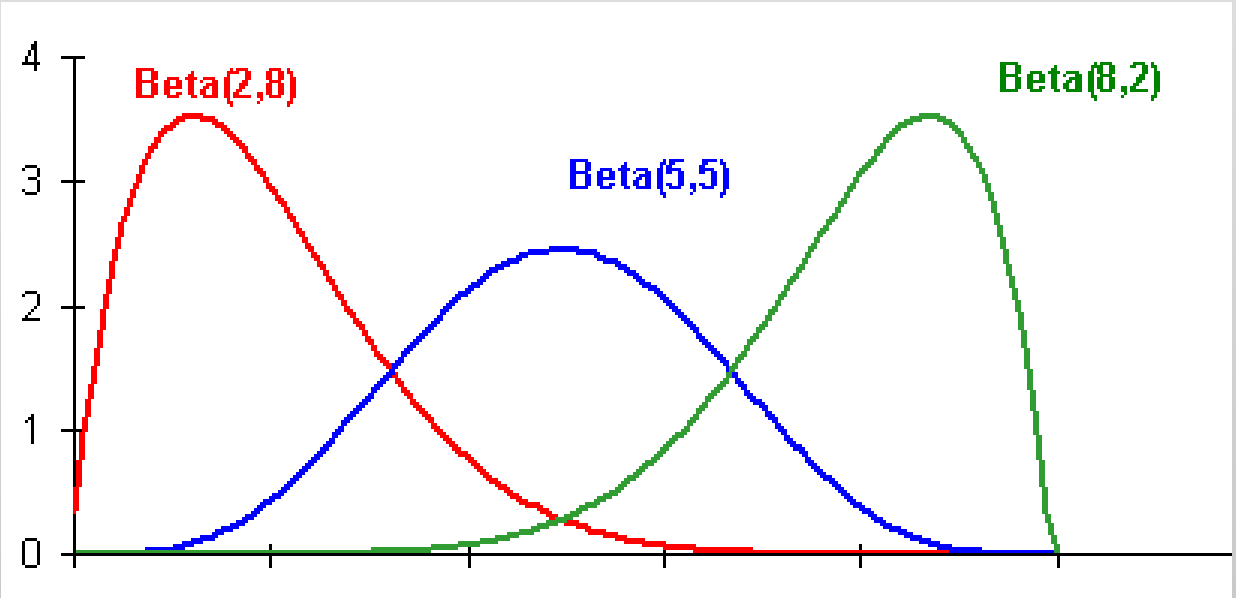


Image: `http://www.vosesoftware.com`

Recall the Basic ABC algorithm

For the observed data $y_{1:n}$, prior $\pi(\theta)$ and distance function ρ:

Algorithm

1. Sample θ^* from prior $\pi(\theta)$
2. Generate $x_{1:n}$ from forward process $f(y \mid \theta^*)$
3. Accept θ^* if $\rho(y_{1:n}, x_{1:n}) < \epsilon$
4. Return to step 1

Generates a sample from an approximation of the posterior:

$$f(x_{1:n} \mid \rho(y_{1:n}, x_{1:n}, \theta) < \epsilon) \cdot \pi(\theta) \approx f(y_{1:n} \mid \theta)\pi(\theta) \propto \pi(\theta \mid y_{1:n})$$

Summary of basic ABC

- Decisions that need to be made:
  1. Distance function (ρ)
  2. Summary statistic(s)
  3. Tolerance (ϵ)
- Finding the "right" ϵ can be inefficient
  $\longrightarrow$ we end up throwing away many of the theories proposed from the selected priors
  $\longrightarrow$ use sequential sampling to improve efficiency

Each exoplanet, i, has the following parameters:

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The "observations"

- The observations are simulated from the forward process (i.e. the hierarchical model)
- Two populations of exoplanets ($N_m = 2$)
- True parameter values:
  1. Standard deviations of h's and k's for each population: $(\sigma_1, \sigma_2) = (0.05, 0.30)$
  2. Weights of each population: $(f_1, f_2) = (.7, .3)$
- $n = 500$ observations

```
for(i in 1:N){
 while(d>epsilon0[t]) {
 #------------------------------------------------------ prior on (f1,f2) ~ Dirichlet(1)
  gamma0 = rgamma(Nm,shape = alpha0 ,scale = 1)
  f.proposed0 = gamma0/sum(gamma0)

 #------------------------------------------------------ prior on (sig1, sig2) ~ Uniform(0,1)
  sigma.proposed0 = runif(Nm)
  index0 = sample.int(Nm, size = n, replace = TRUE, prob = f.proposed0)
  sigma.proposed = sigma.proposed0[index0]

 #------------------------------------------------------ generate the h's and k's
  theta1.h = sigma.proposed*rnorm(n)
  theta1.k = sigma.proposed*rnorm(n)

 #------------------------------------------------------ add measurement error
  theta1.hhat = theta1.h + sigma.hhat*rnorm(n)
  theta1.khat = theta1.k + sigma.khat*rnorm(n)

  #------------------------------------------------------ check the distance
  x[,1] = theta1.hhat
  x[,2] = theta1.khat
  d = distance.function(x, data0)
   }

  #------------------------------------------------------ if distance <= epsilon, then keep
  d0[i,t] = d
  sigma.final[i, ,t]= sigma.proposed0
  f.final[i,,t]= f.proposed0
  gamma.sum[i,t] = sum(gamma0)
    }
```

Draw from the prior

```
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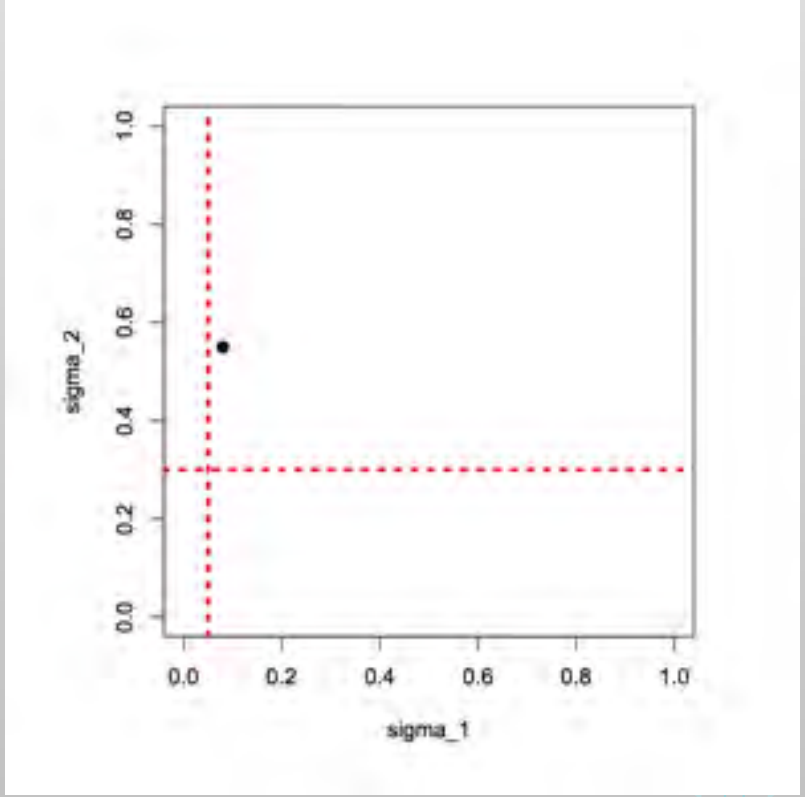

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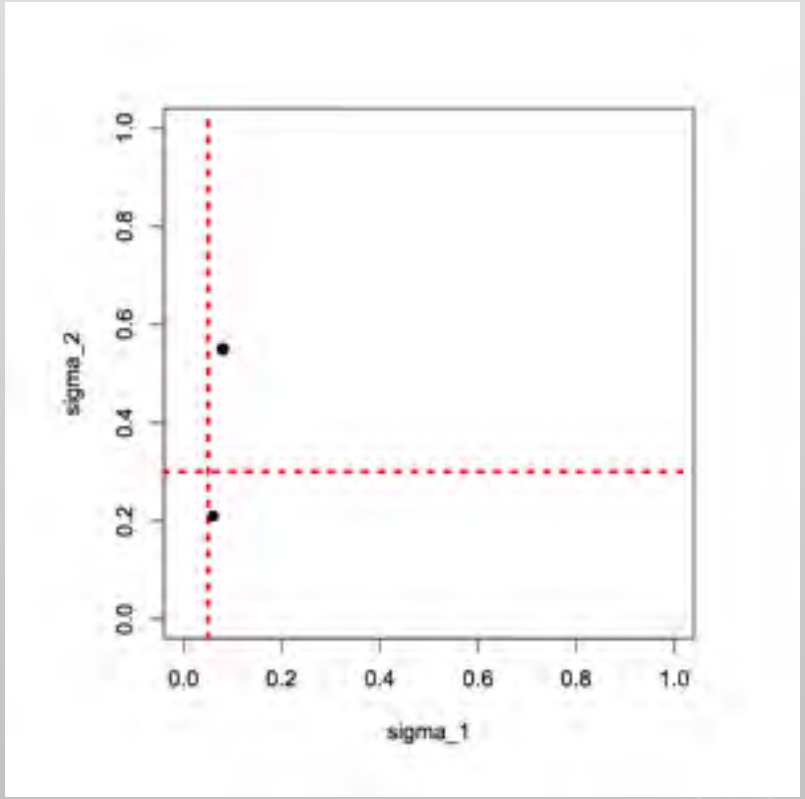

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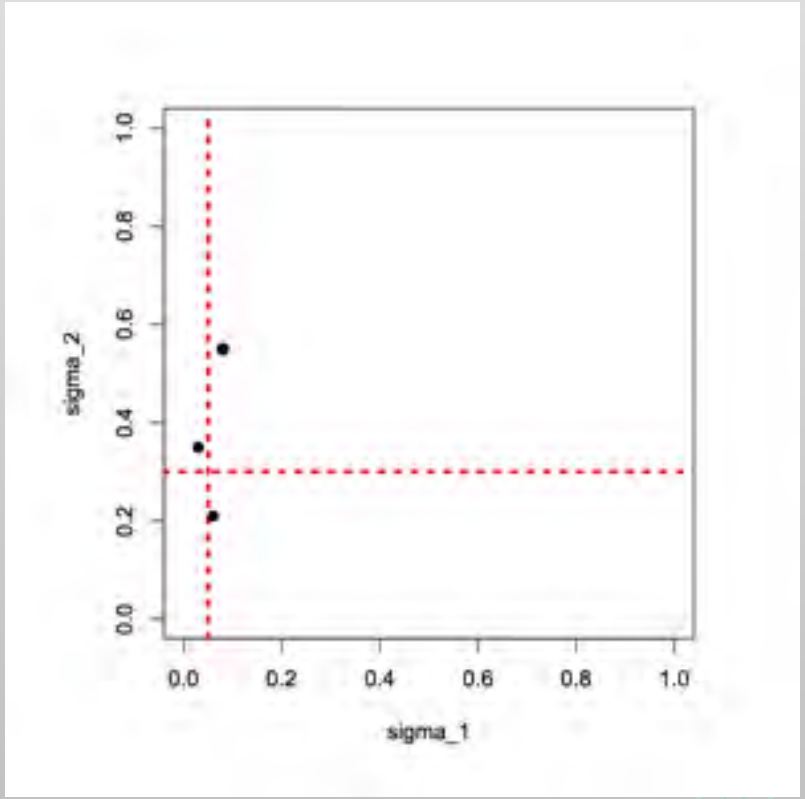


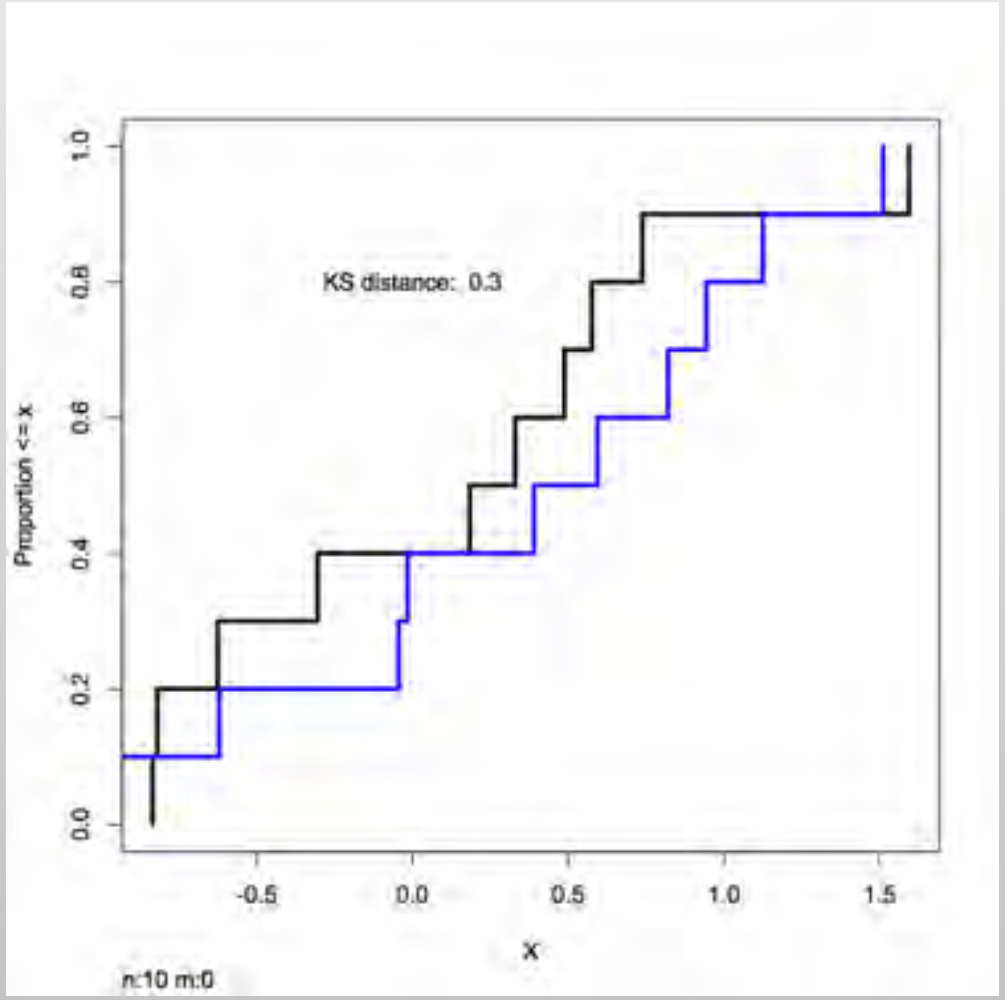
Forward model

```
#------------------------------ generate the h's and k's
 theta1.h = sigma.proposed*rnorm(n)
 theta1.k = sigma.proposed*rnorm(n)

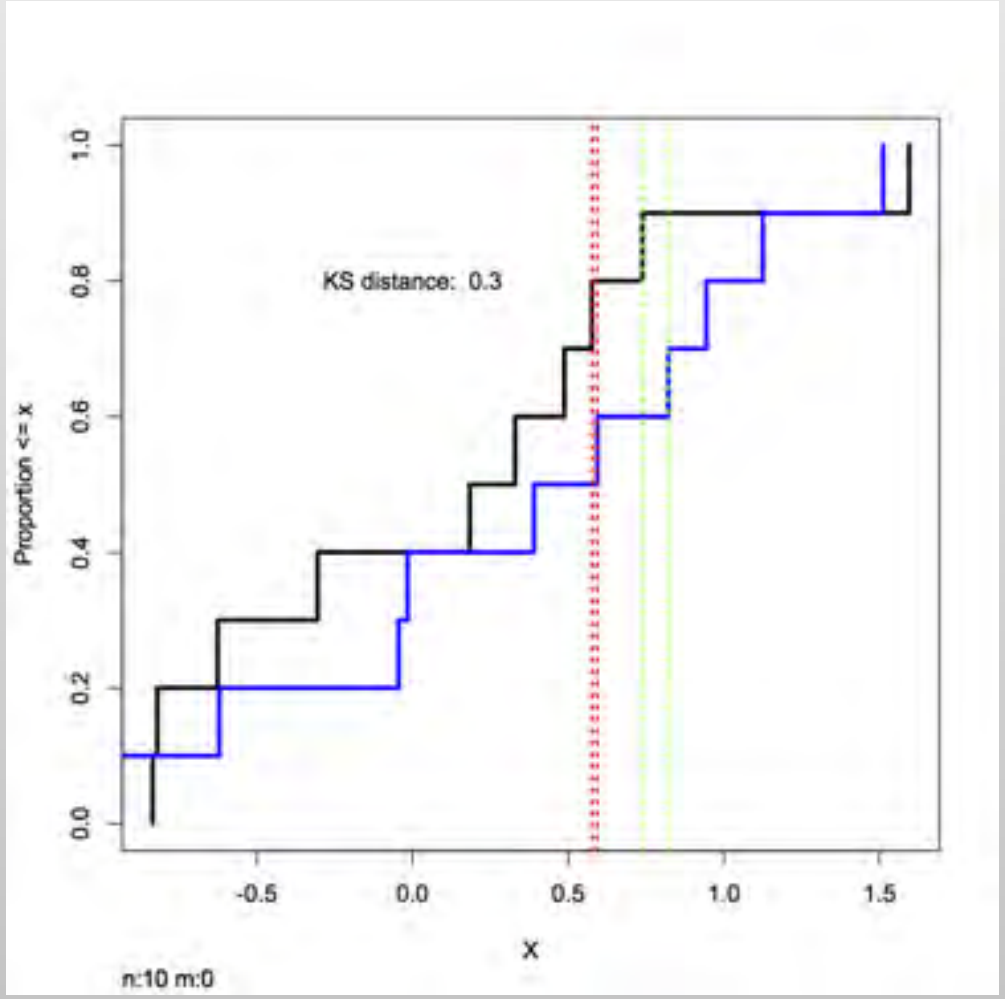

#------------------------------  add measurement error
 theta1.hhat = theta1.h + sigma.hhat*rnorm(n)
 theta1.khat = theta1.k + sigma.khat*rnorm(n)


#------------------------------ check the distance
 x[,1] = theta1.hhat
 x[,2] = theta1.khat
 d = distance.function(x, data0)
   }
```

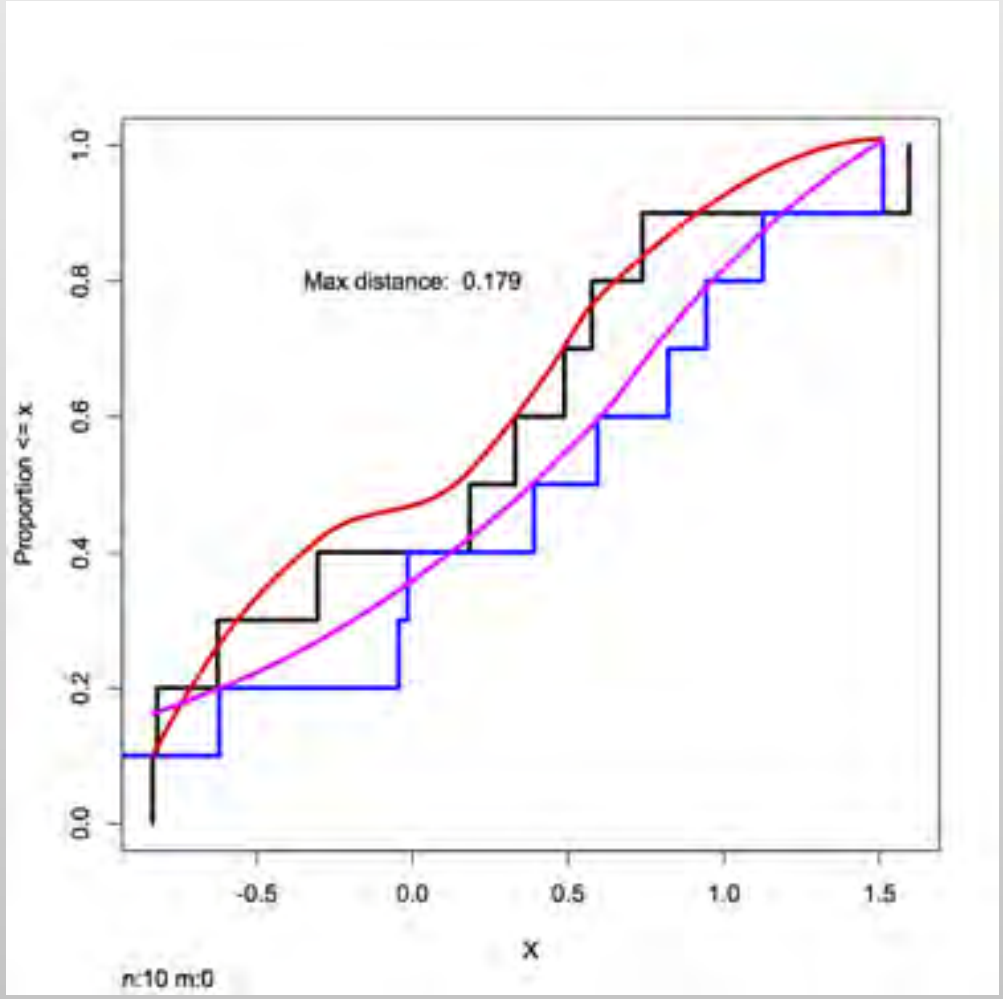
Distance function: Kolmogorov-Smirnov distance



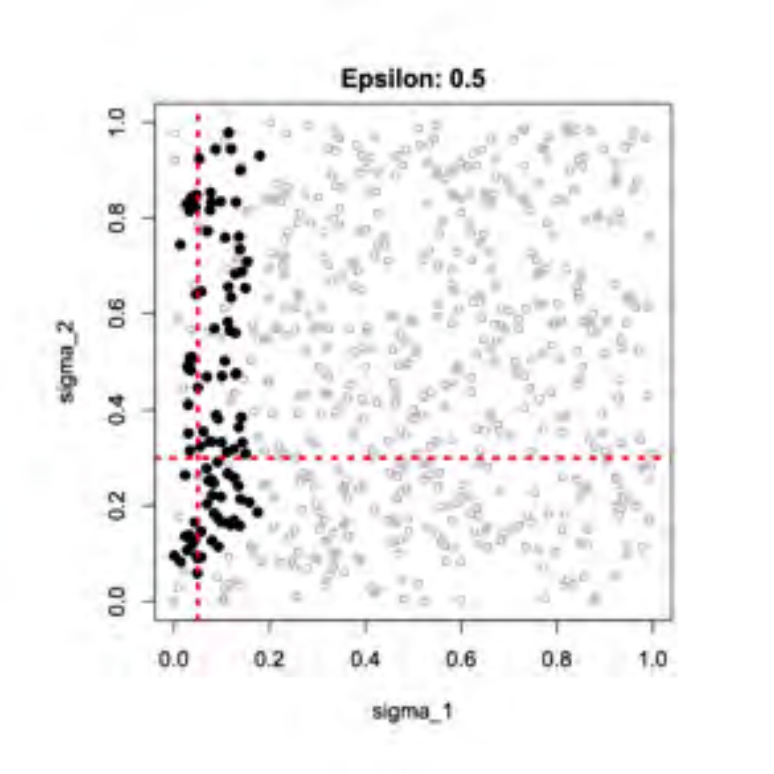
Distance function: Kolmogorov-Smirnov distance

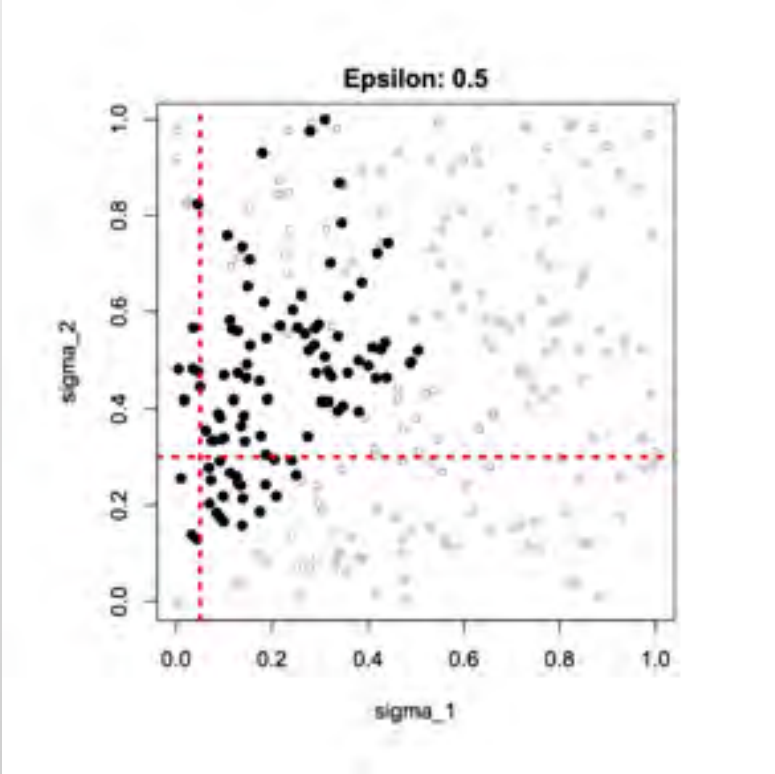


Distance function: Kolmogorov-Smirnov distance

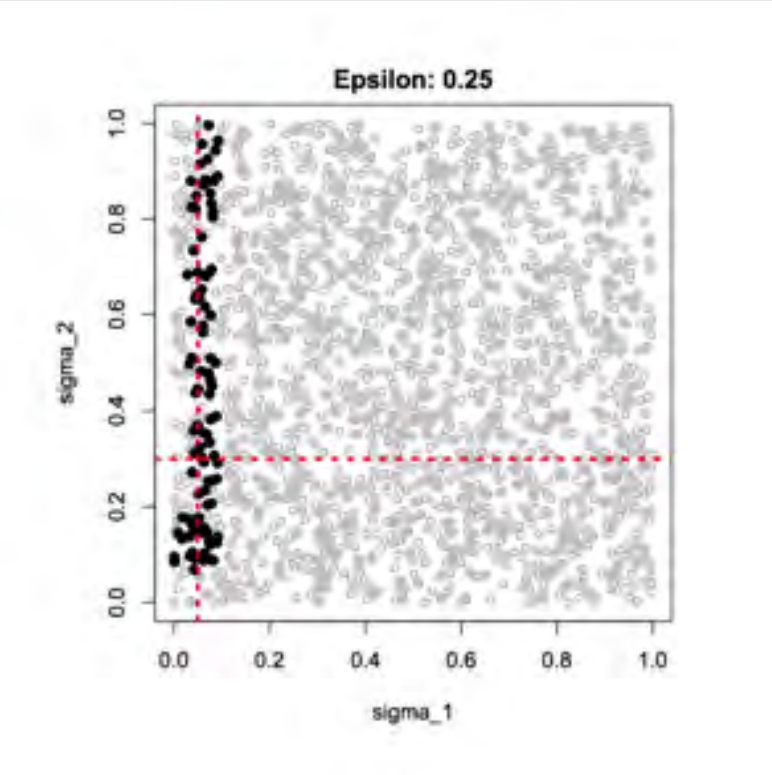


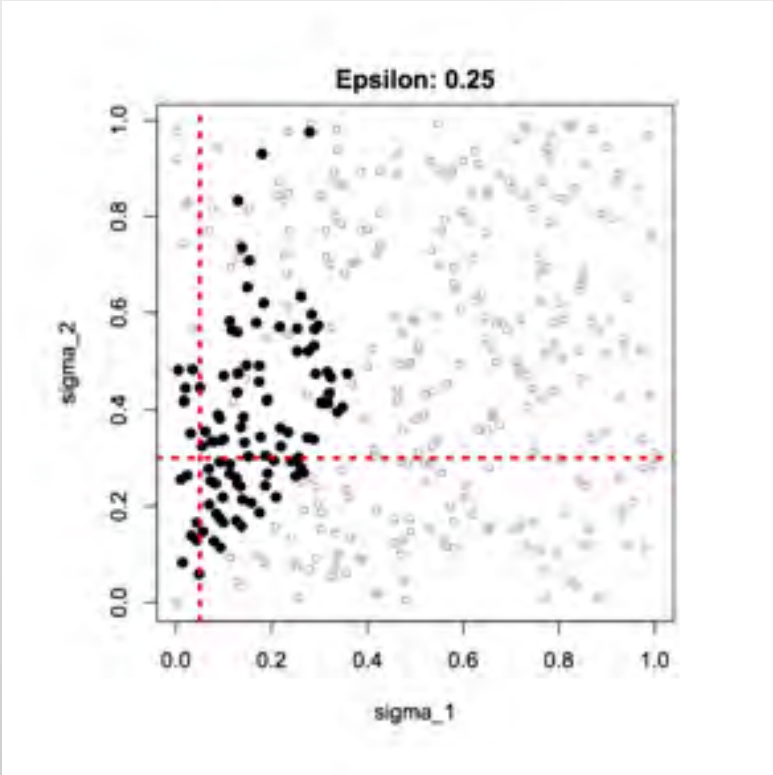
Does the distance function really matter?





- KS distance on e^2 (left); mean difference of e^2 (right)
- Drawing 100 particles





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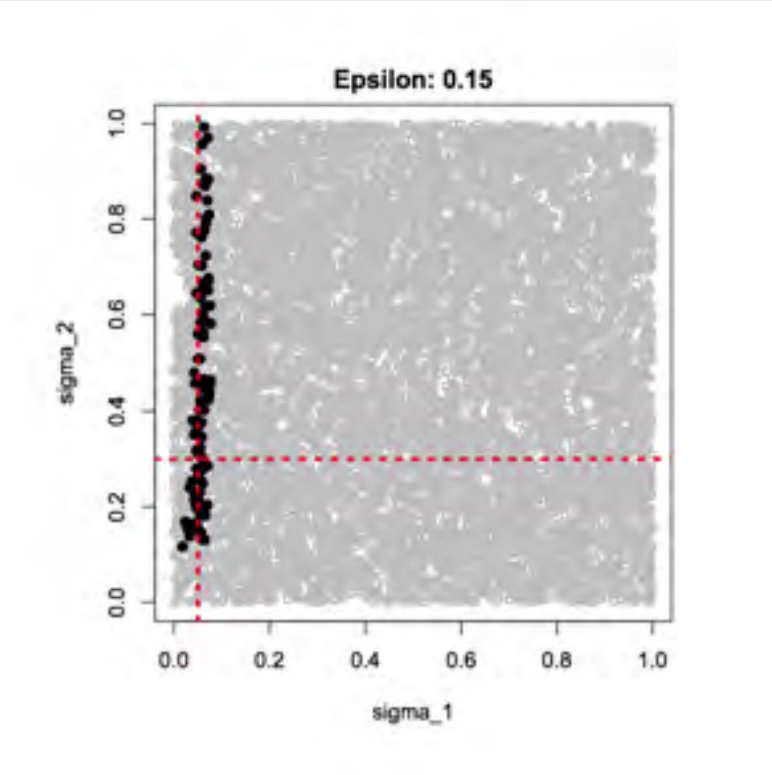

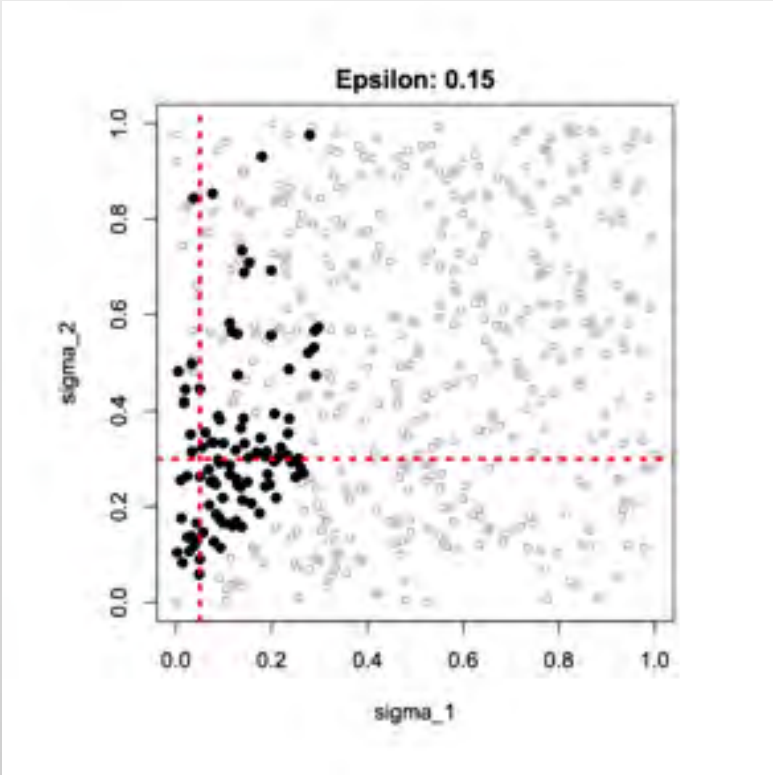

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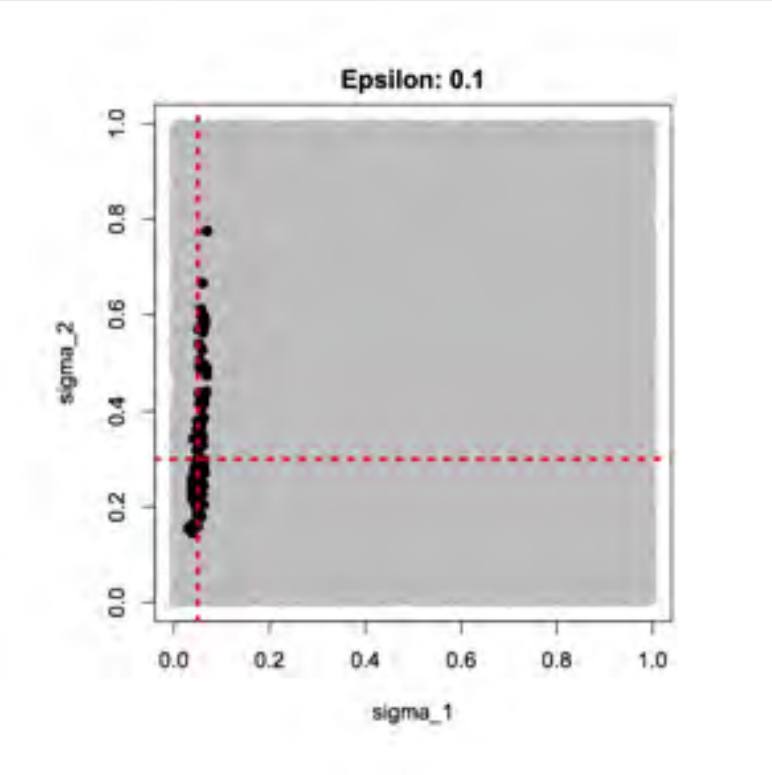
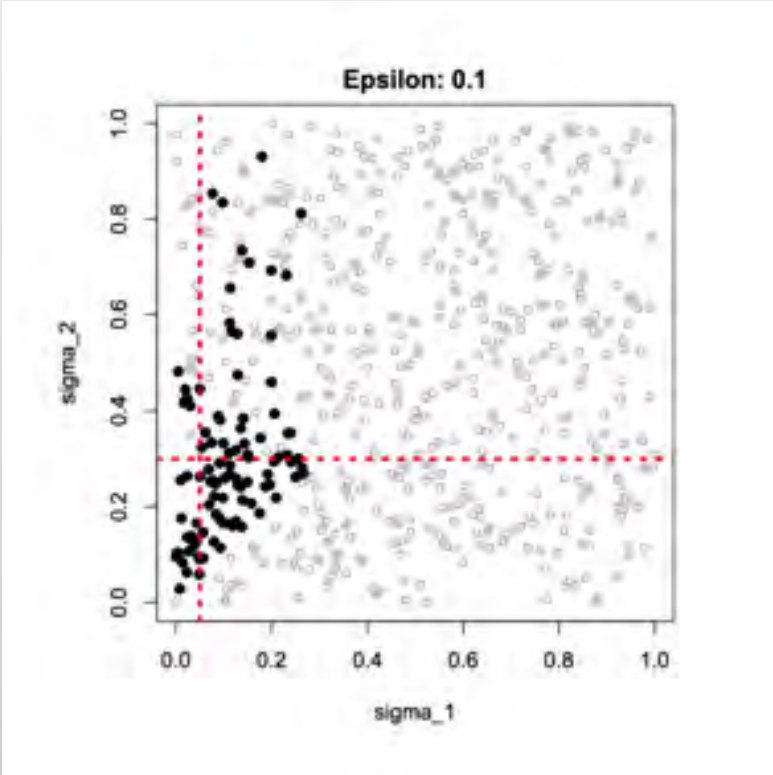
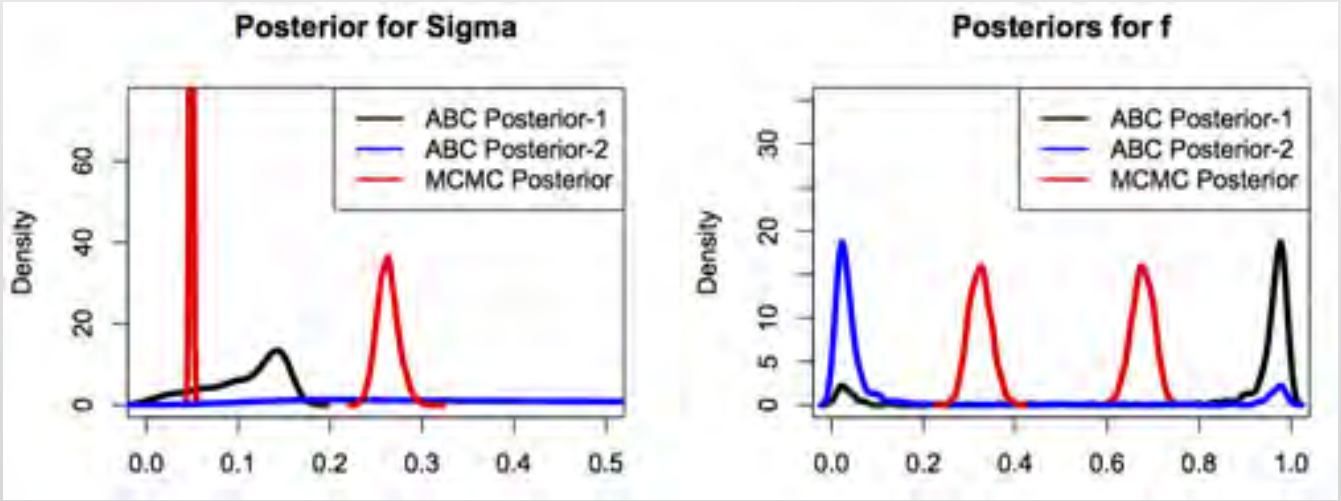




- KS distance on e^2 (left); mean difference of e^2 (right)
- Drawing 100 particles

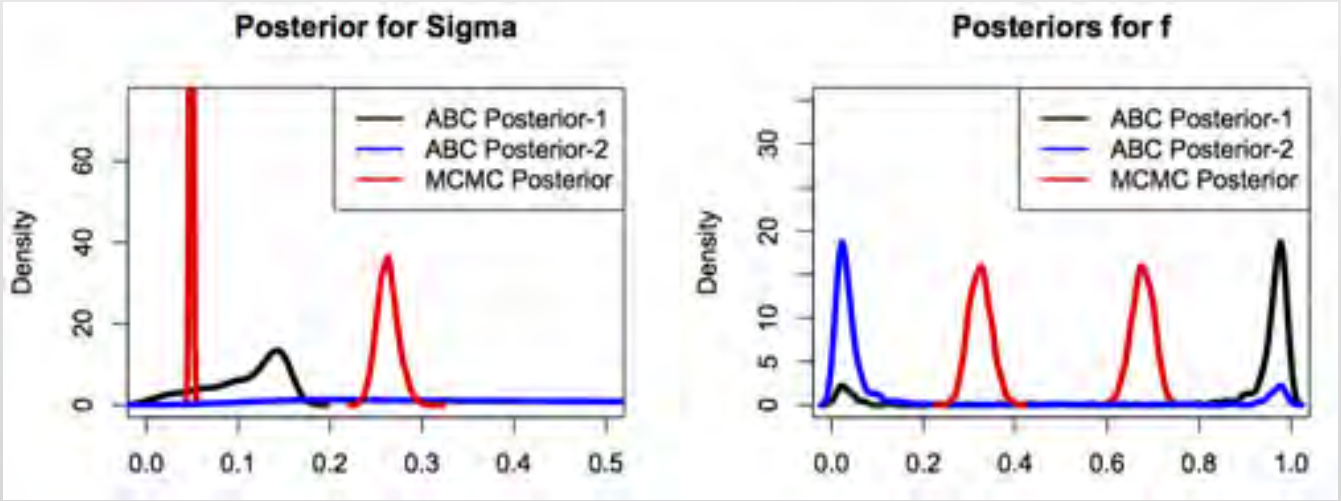
$N = 500$ particles

Mean difference of e^2

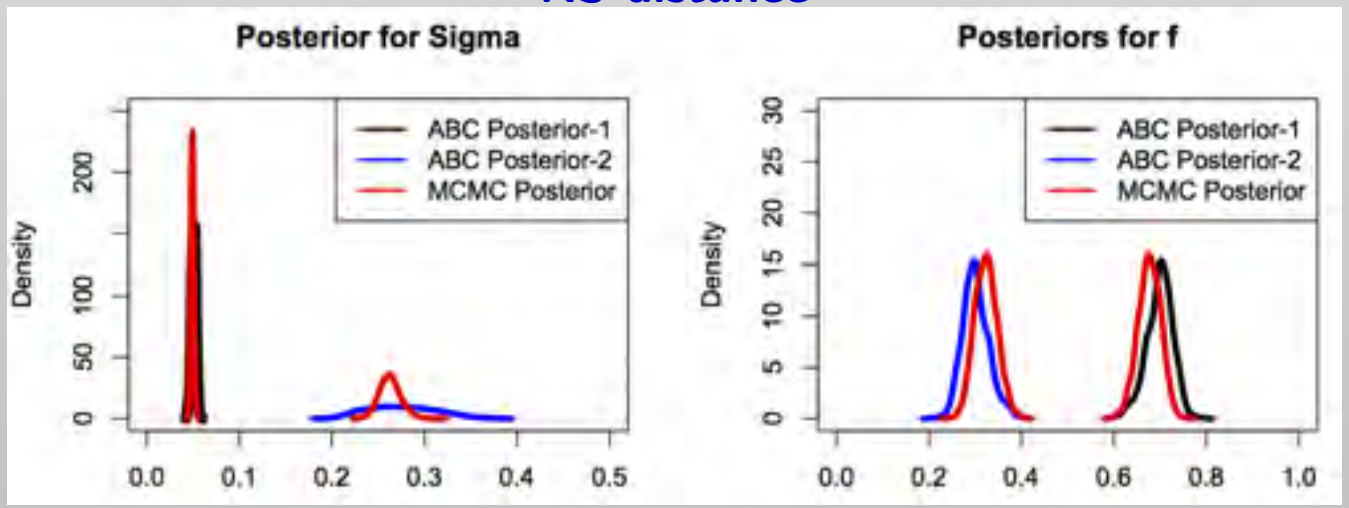


$N = 500$ particles

Mean difference of e^2



KS distance



Decreasing tolerances $\epsilon_1 \geq \cdots \geq \epsilon_T$

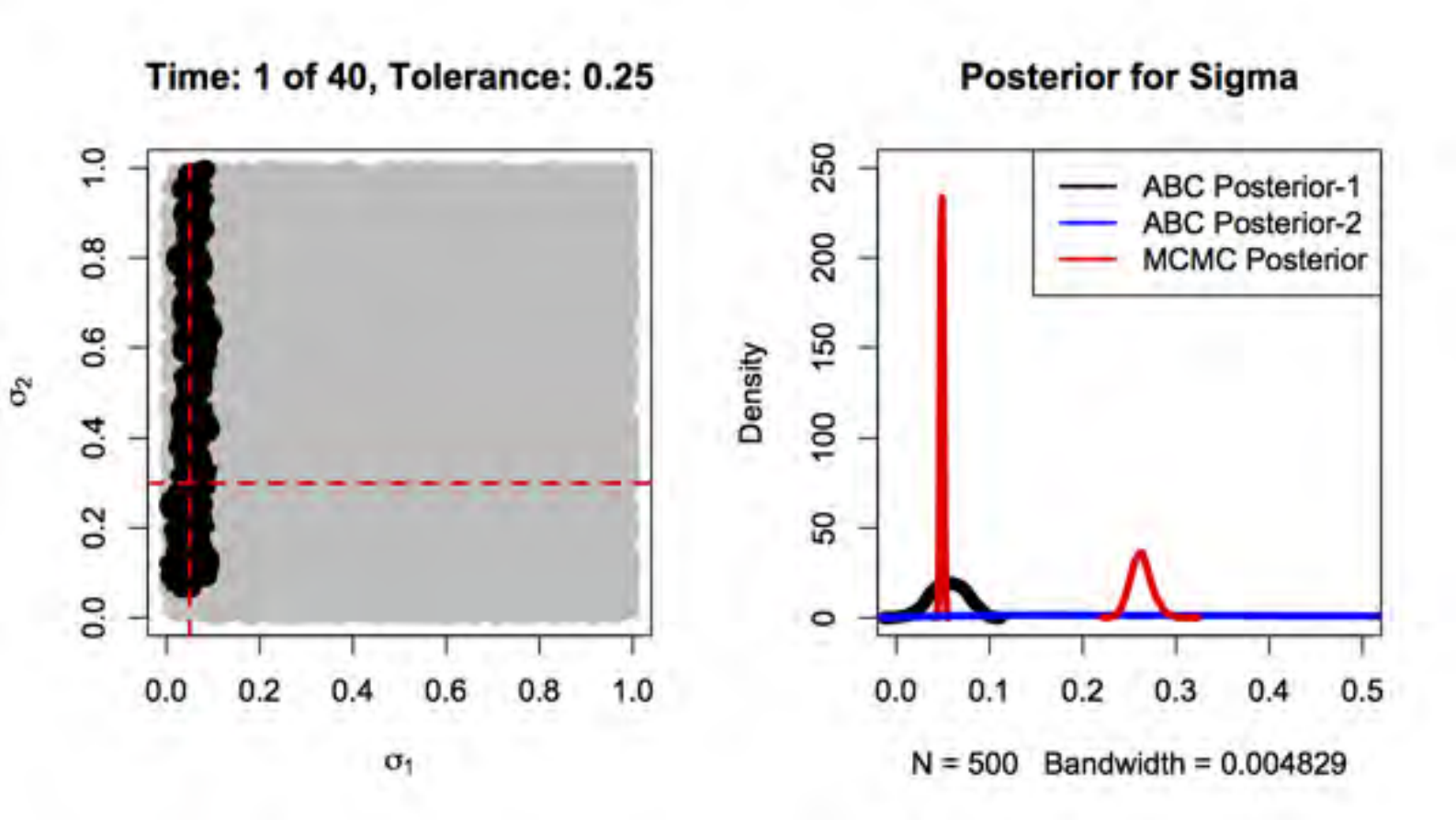
ABC - Population Monte Carlo algorithm* (ABC - PMC)

1. At $t = 1$
   For $i = 1, \ldots, N$ particles
   Generate $\theta_i^{(1)} \sim \pi(\theta)$ and $x \sim f(y \mid \theta_i^1)$ until $\rho(y, x) < \epsilon_1$
   Set $w_i^{(1)} = N^{-1}$
2. At $t = 2, \ldots, T$
   Set $\tau_t^2 = 2 \cdot \text{var}\left(\theta_{1:N}^{(t-1)}\right)$
   For $i = 1, \ldots, N$ particles
   Draw $\theta_i^* \sim \text{multinomial}\left(\theta_{1:N}^{(t-1)}, w_{1:N}^{(t-1)}\right)$
   Generate $\theta_i^{(t)} \mid \theta_i^* \sim N(\theta_i^*, \tau_t^2)$ and $x \sim f(y \mid \theta_i^{(t)})$ until $\rho(y, x) < \epsilon_t$
   Set $w_i^{(t)} \propto \pi(\theta_i^{(t)}) / \sum_{j=1}^N w_j^{(t-1)} \phi[\tau_t^{-1}(\theta_i^{(t)} - \theta_j^{(t-1)})]$

- $\phi(\cdot)$ is the density function of a $N(0, 1)$

*From Beaumont et al. (2009)

KS -distance with N = 500 particles



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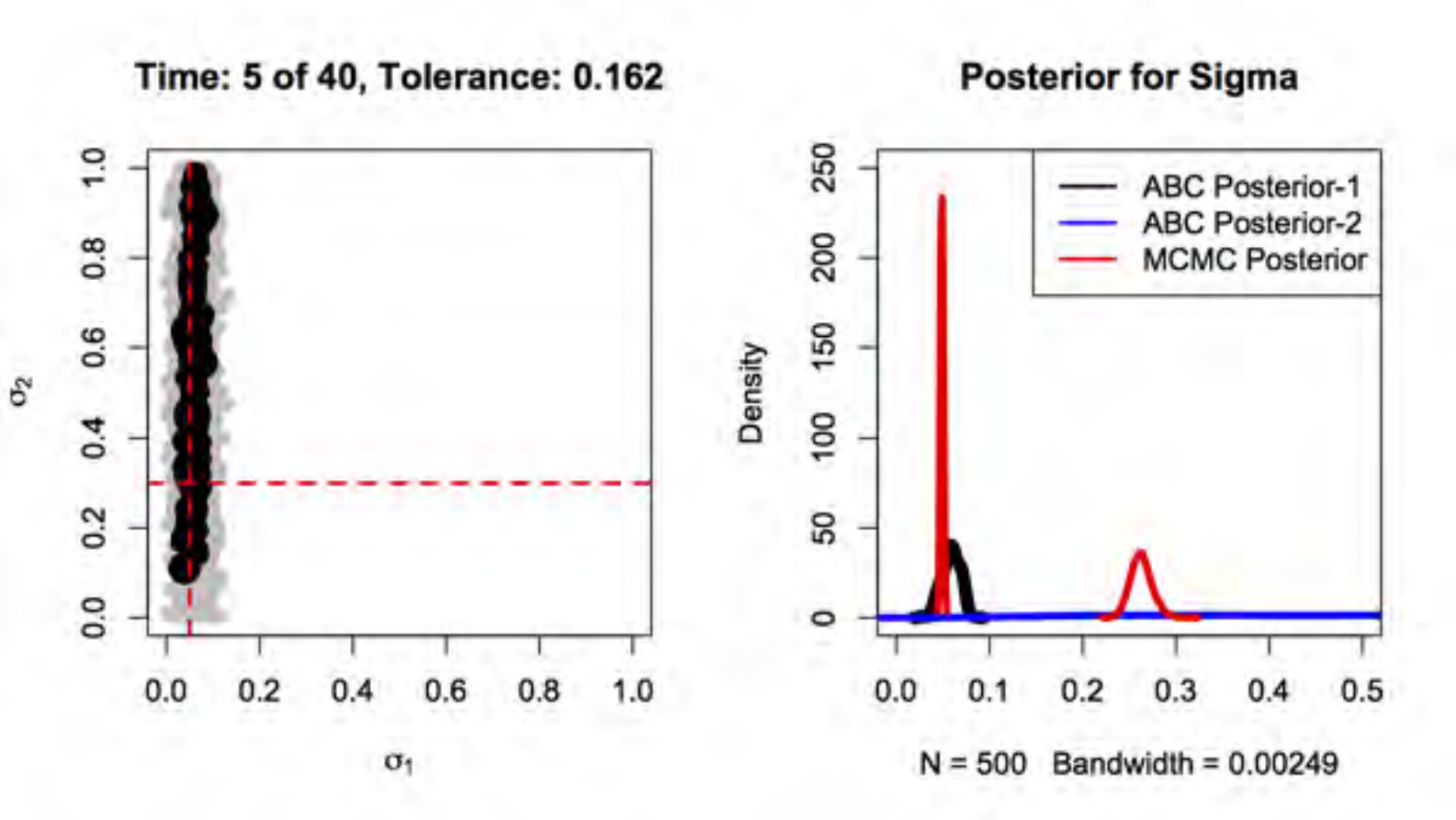

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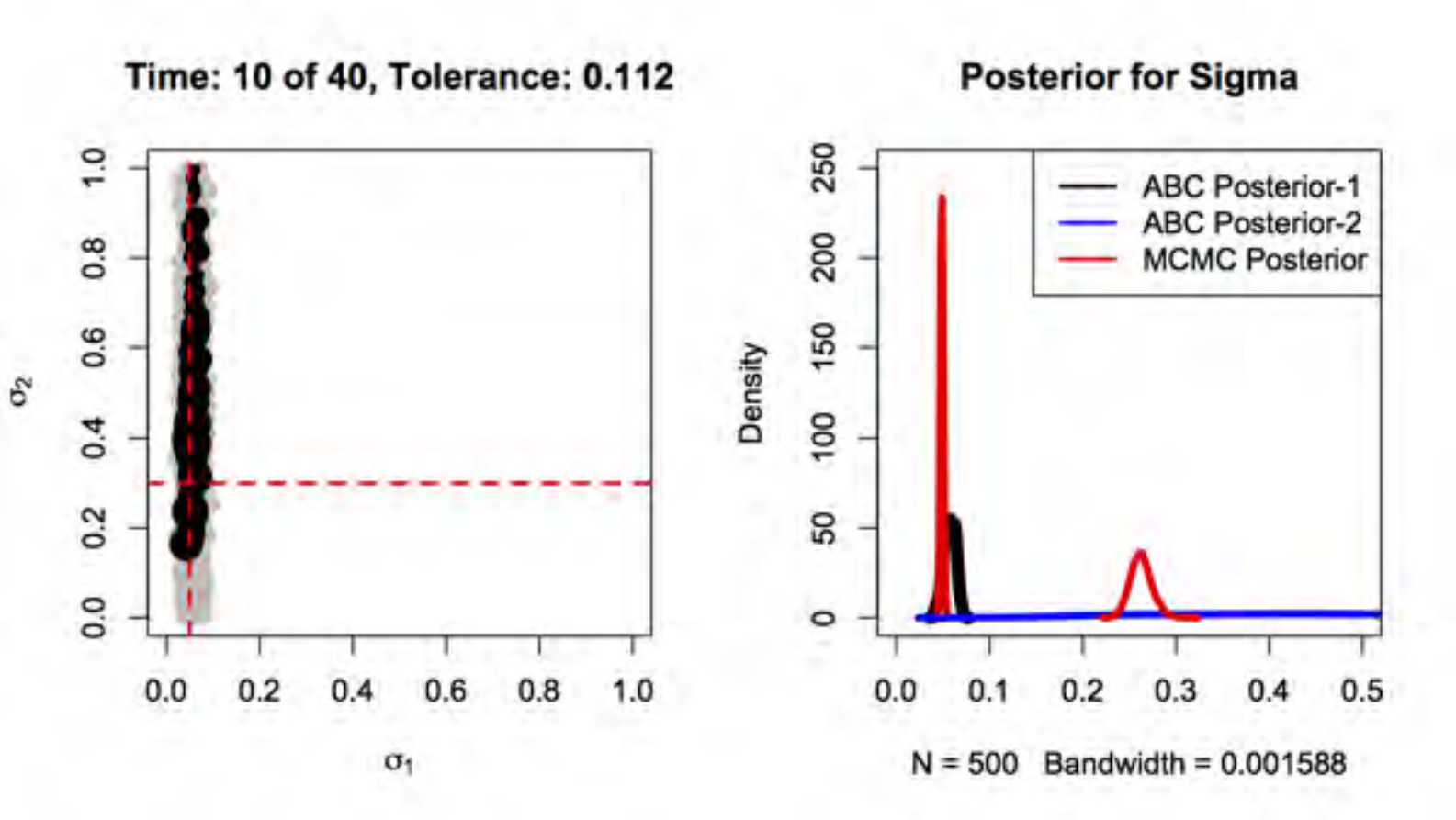

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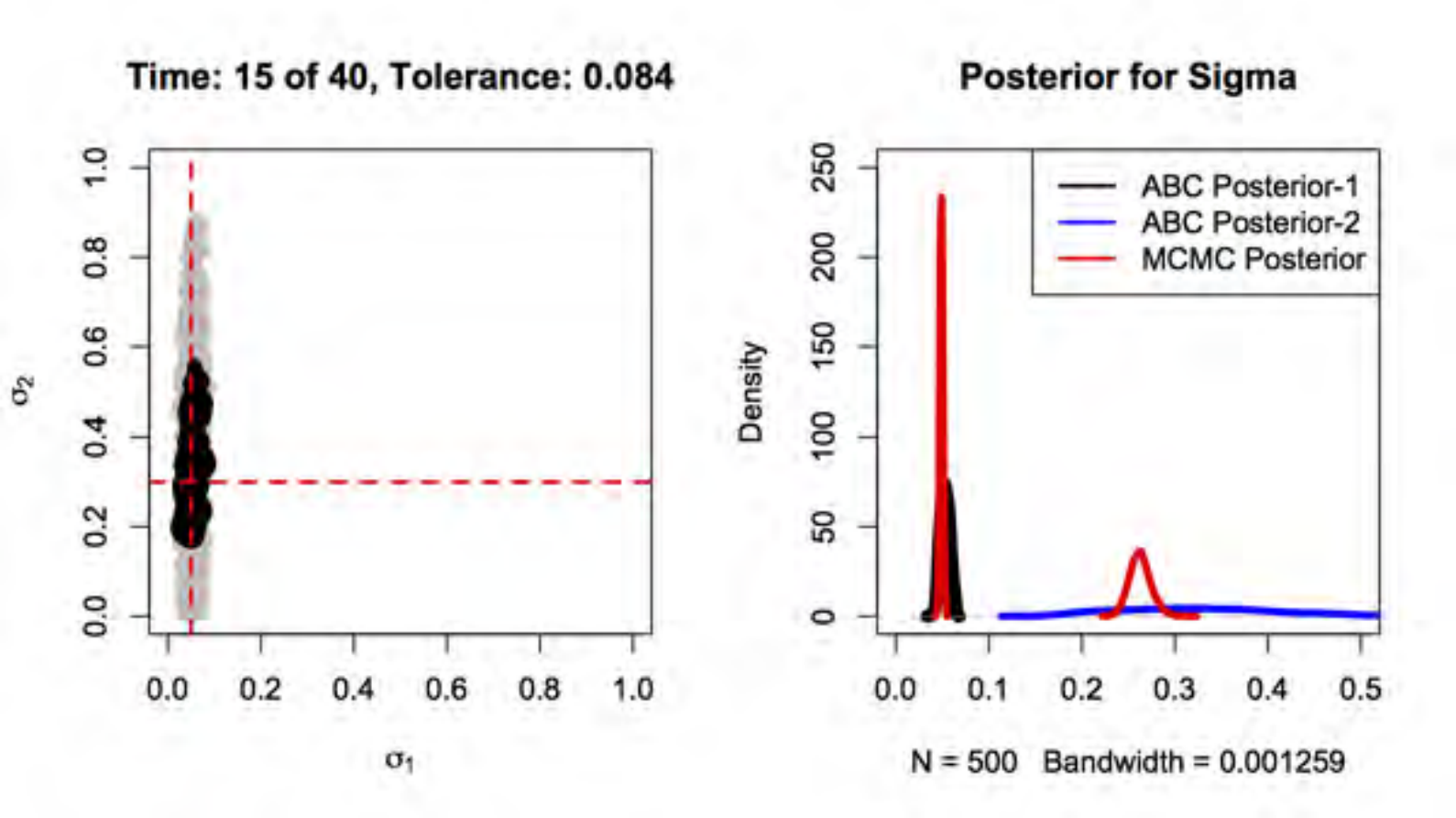

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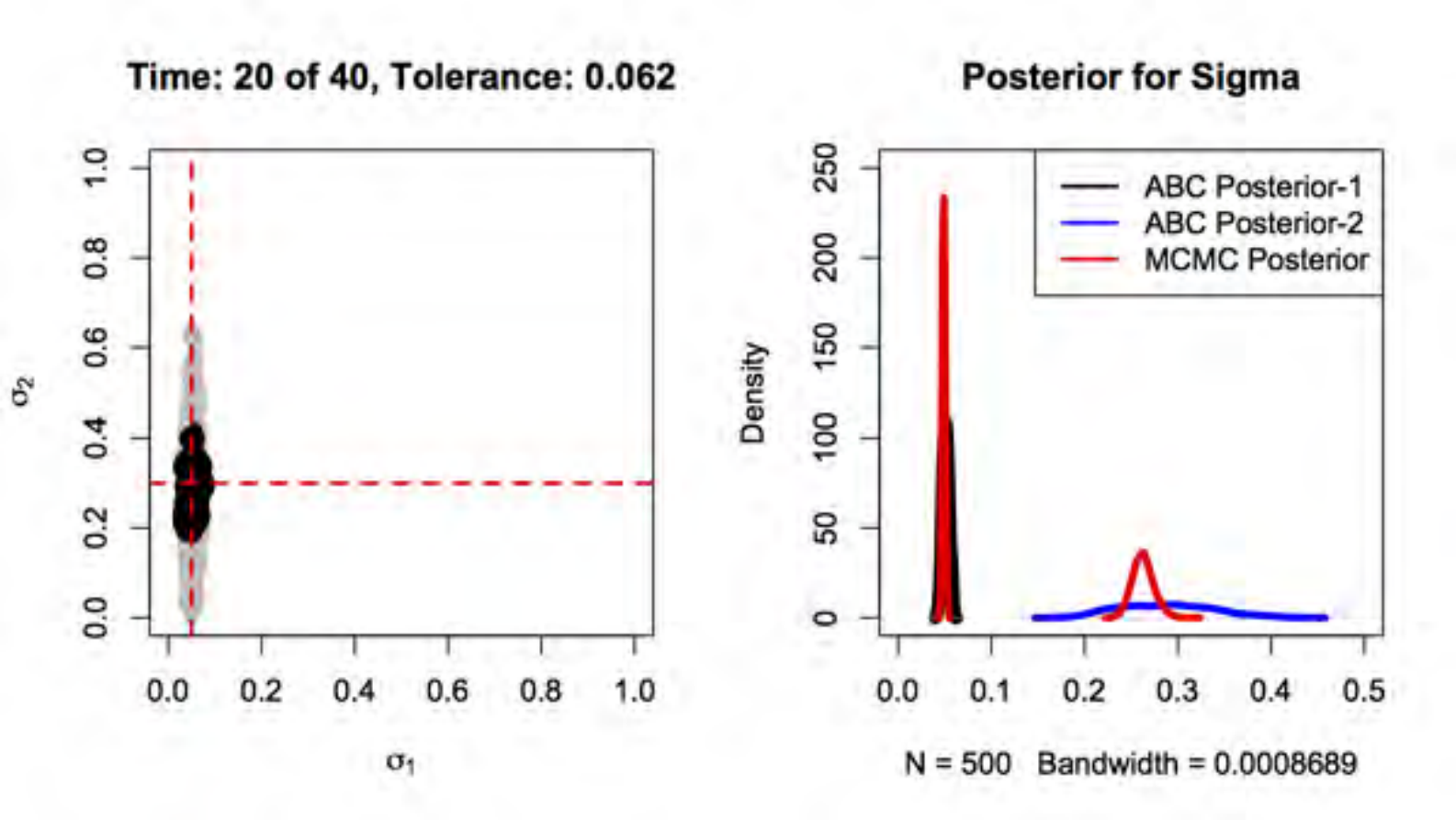

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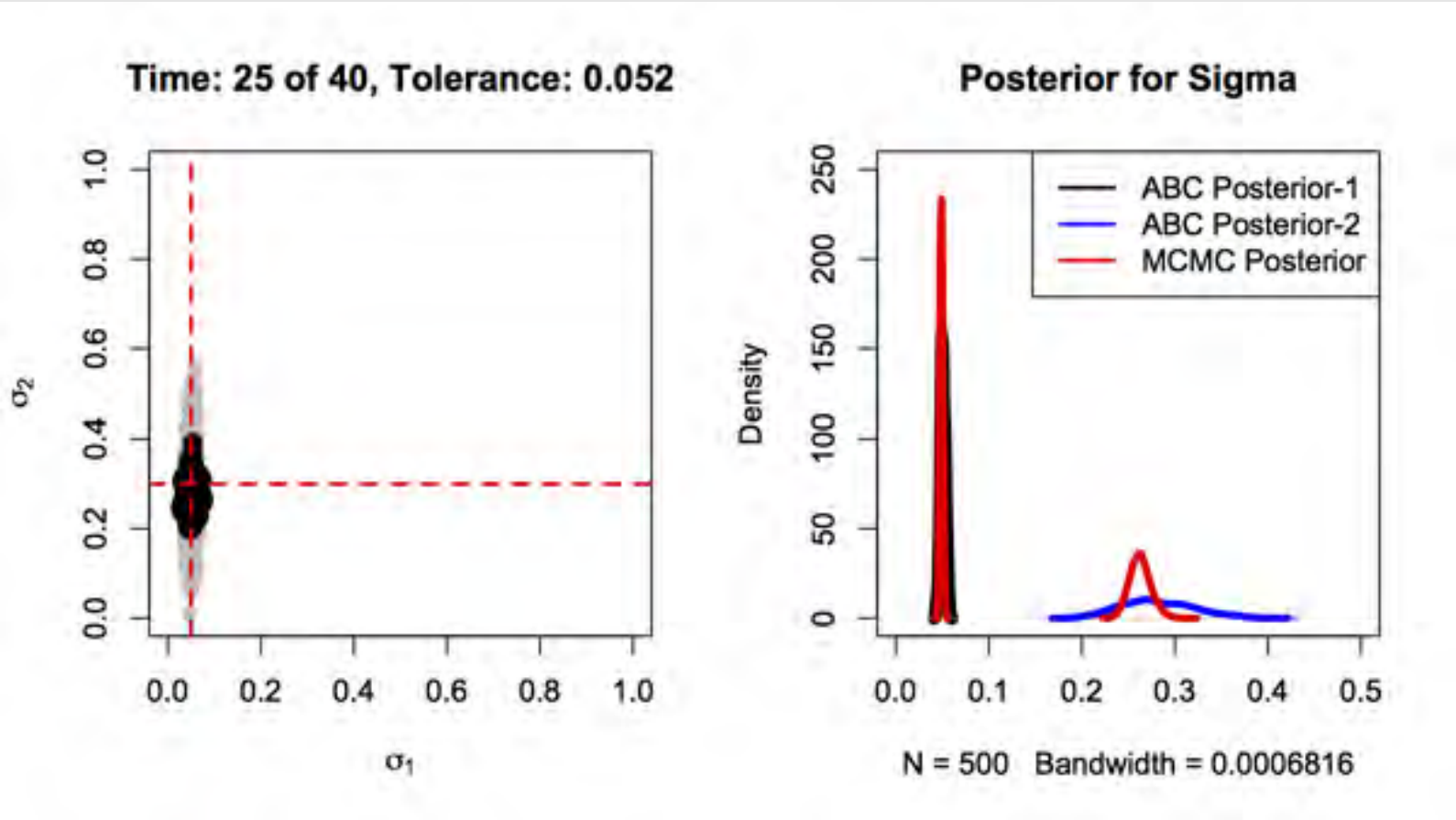

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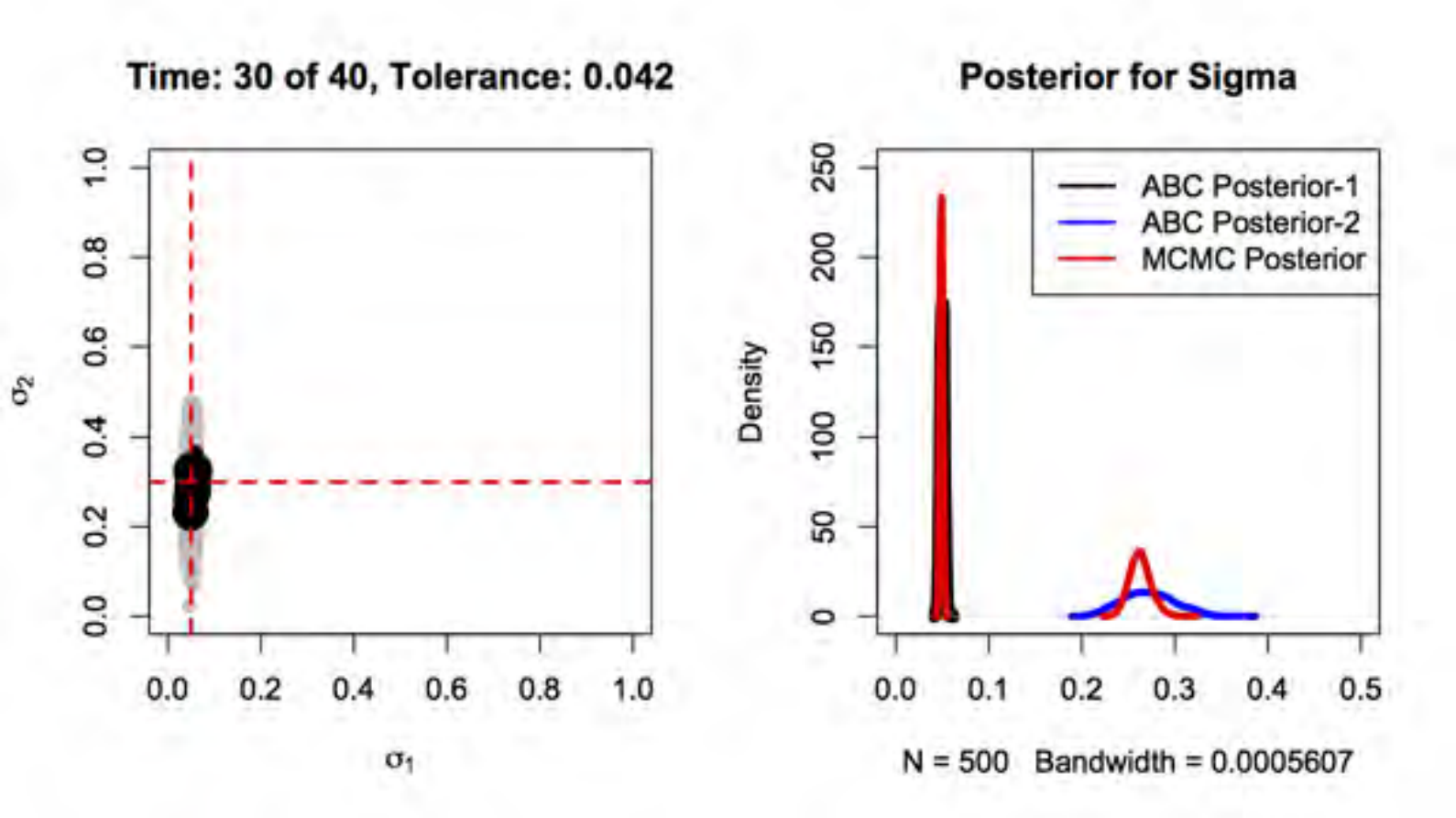

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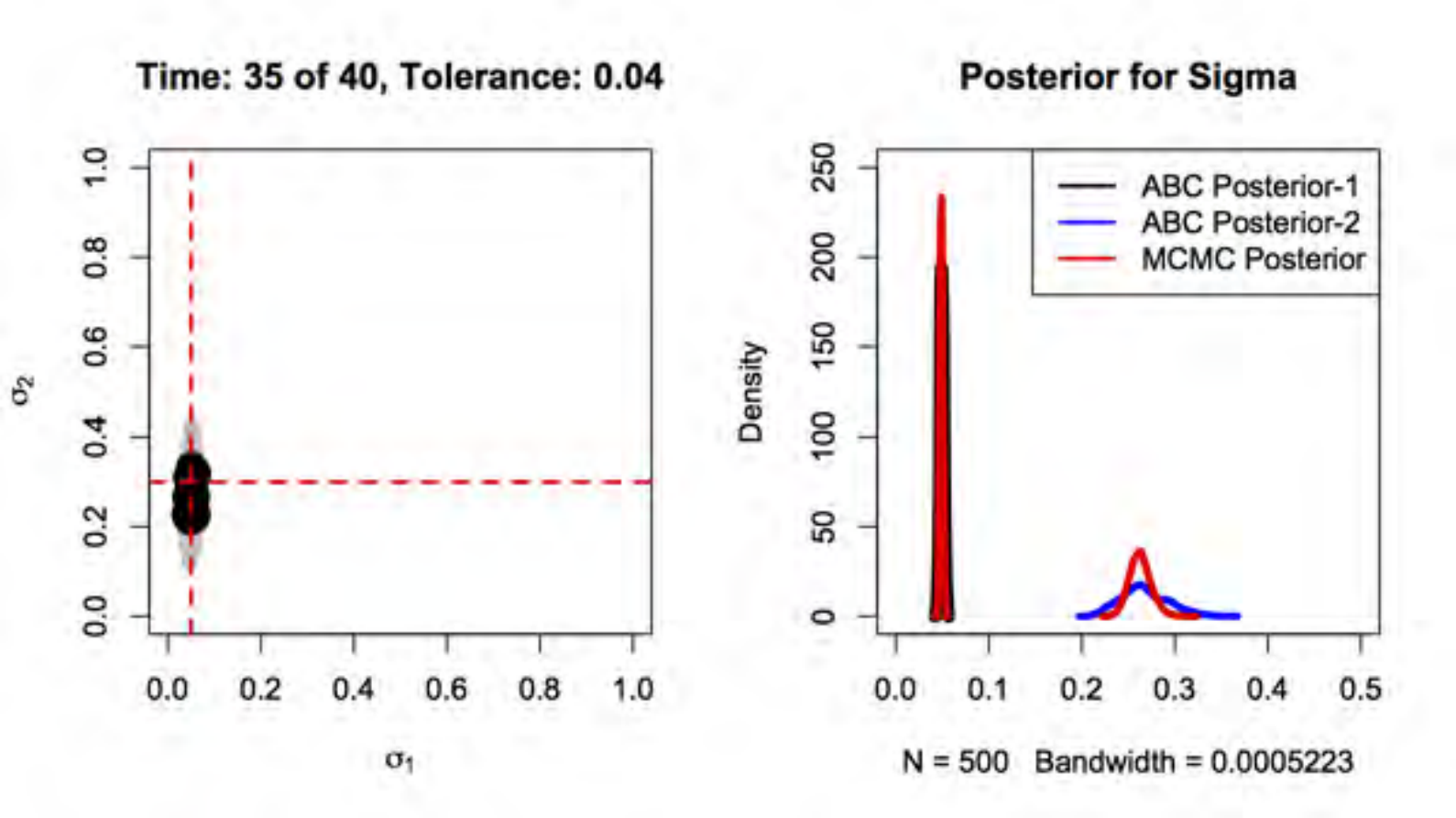

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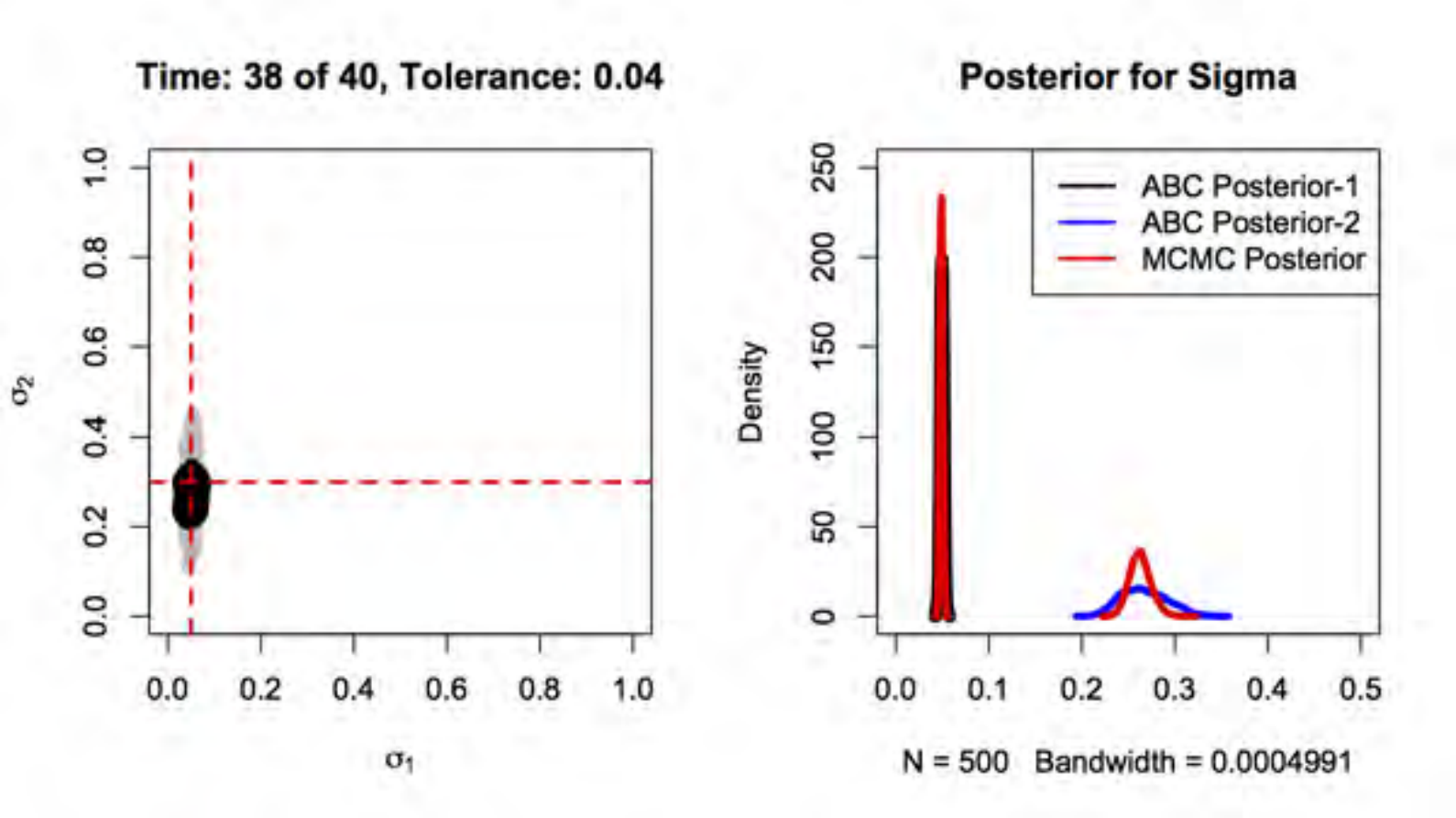

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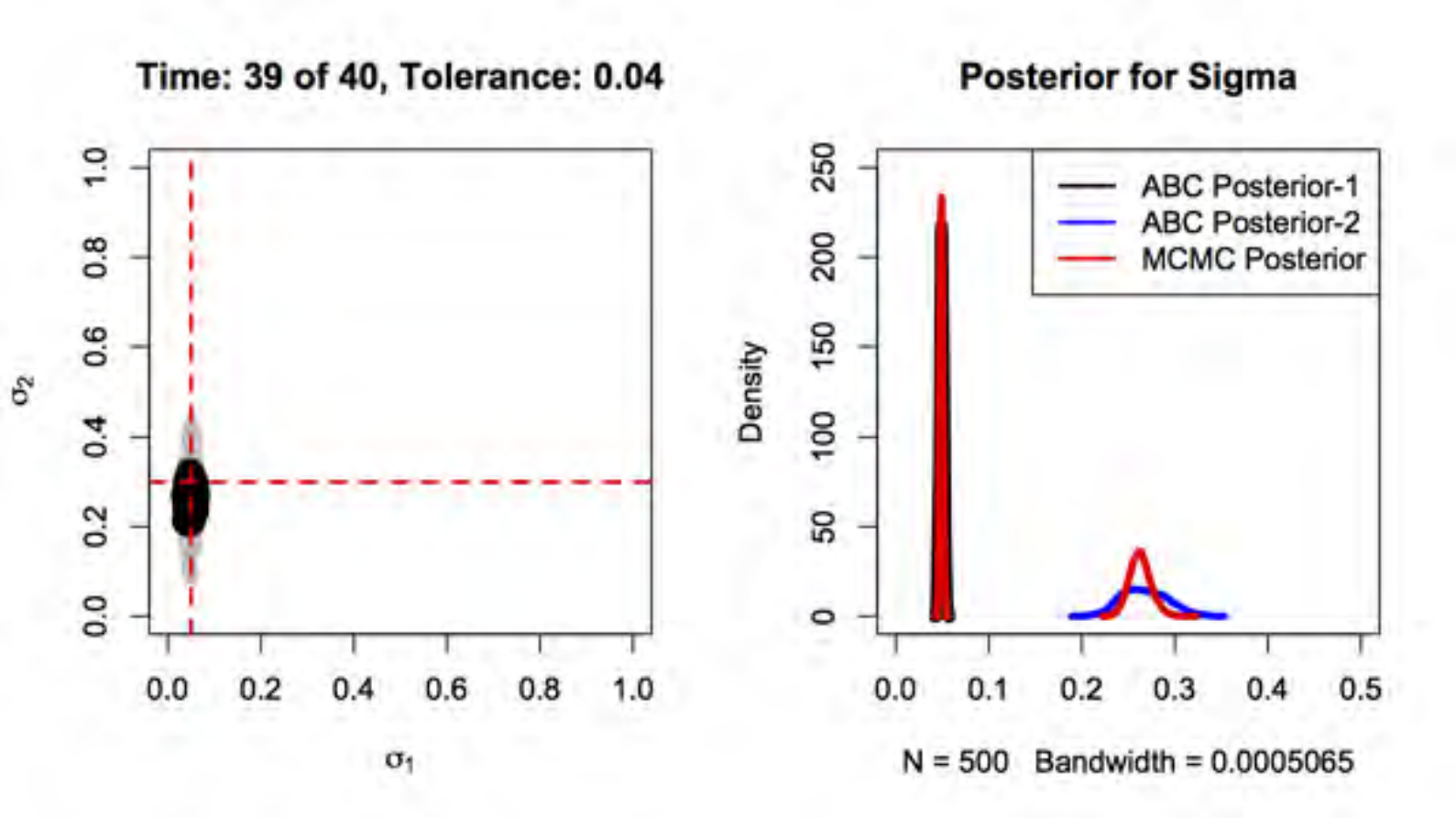
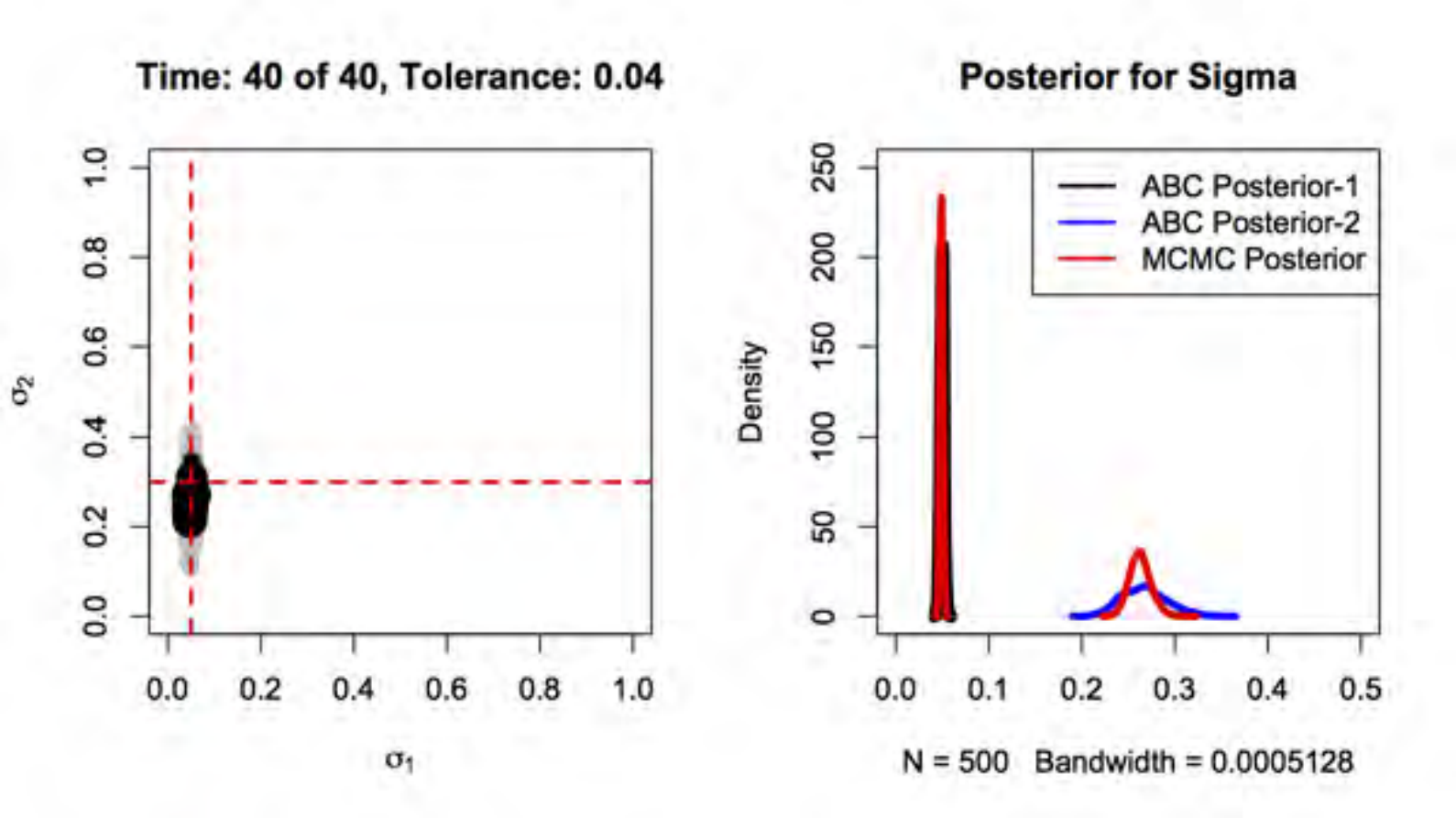
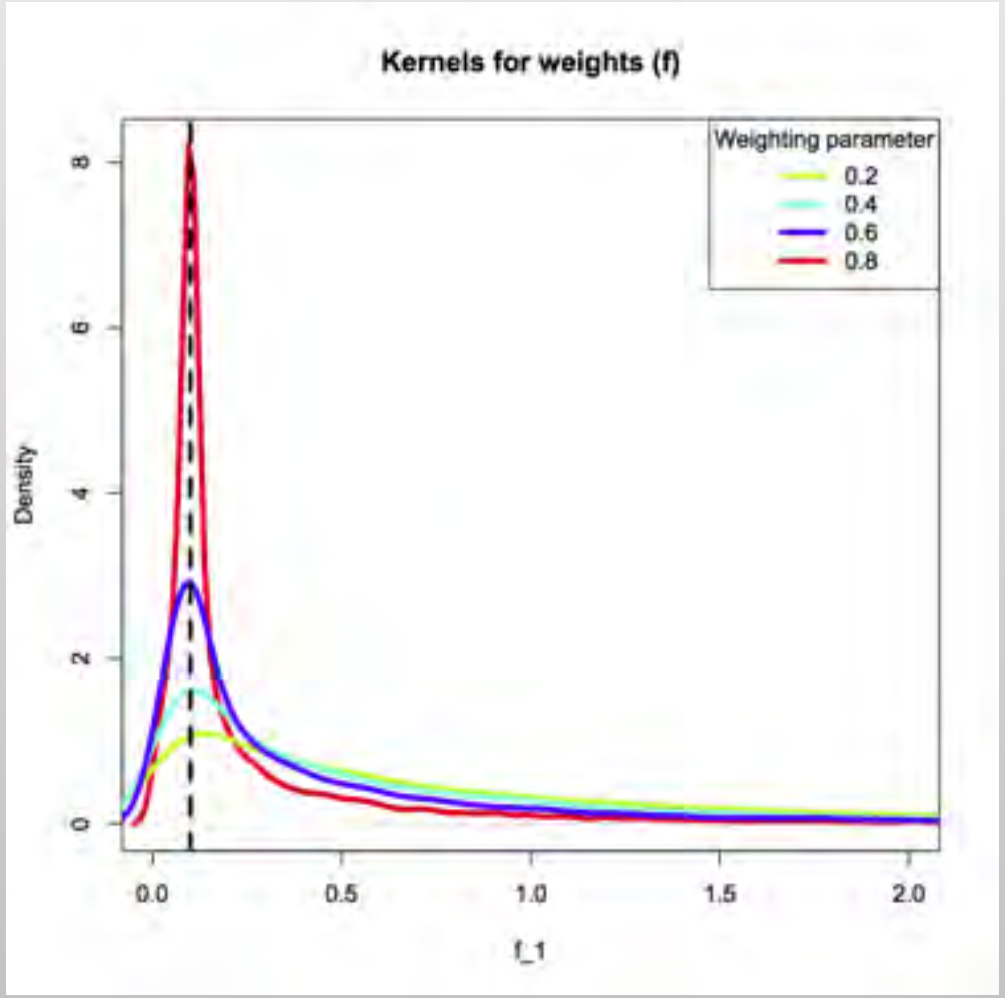


KS -distance with N = 500 particles



Transition kernels.

Moving particles: $\left\{ f_1^{(J)}, \ldots, f_{Nm}^{(J)} \right\}_{J=1}^{N}$



Moving particles: $\left\{ f_1^{(J)}, \ldots, f_{Nm}^{(J)} \right\}_{J=1}^{N}$

$$f_{old} = (f_1, f_2, \ldots, f_{Nm}) \text{ (original mixture weights)}$$

$$f_{old} \sim Dir(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{Nm}))$$

$$\xi_i \sim Gamma(\alpha_i, 1)$$

$$\implies \xi_+ = \xi_1 + \cdots + \xi_{Nm} \sim Gamma(\alpha_+ = \sum \alpha_i, 1)$$

$$f_{old} = \left(\frac{\xi_1}{\xi_+}, \ldots, \frac{\xi_{Nm}}{\xi_+} \right) \sim Dir(\alpha)$$

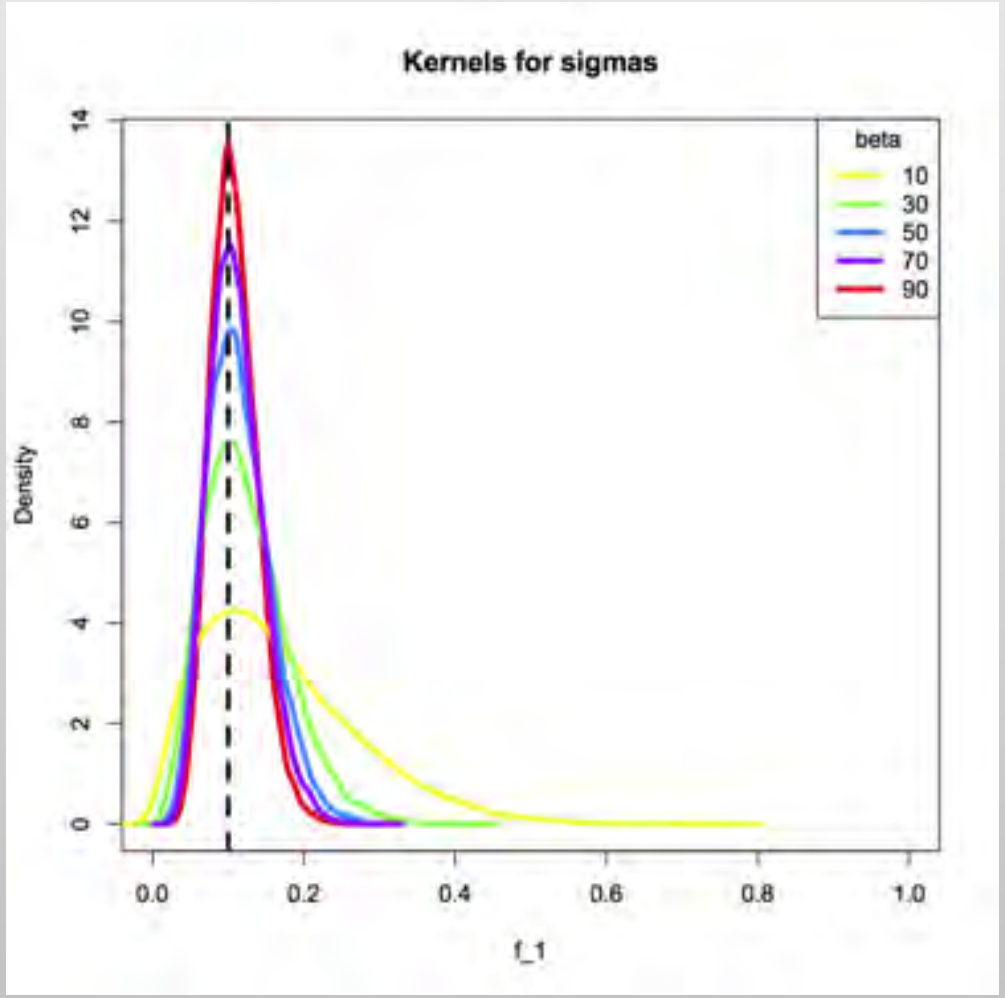
Moving particles: $\left\{ f_1^{(J)}, \ldots, f_{Nm}^{(J)} \right\}_{J=1}^{N}$

Given weighting parameter p

$$
\begin{aligned}
B_i &\sim \text{Beta}(p\alpha_i, (1-p)\alpha_i) \\
\zeta_i &\sim \text{Gamma}((1-p)\alpha_i, 1) \\
\xi_i^* &= \xi_i B_i + \zeta_i, i = 1, \ldots, N_m \\
&\implies \xi_i^* \sim \text{Gamma}(\alpha_i, 1) \\
&\implies \left(\frac{\xi_1^*}{\xi_+^*}, \ldots, \frac{\xi_{Nm}^*}{\xi_+^*} \right) \sim Dir(\alpha)
\end{aligned}
$$

where $\xi_+^* = \sum_{i=1}^{Nm} \xi_i^*$.

Moving particles: $\left\{\sigma_1^{(J)}, \ldots, \sigma_{Nm}^{(J)}\right\}_{J=1}^{N}$



Recall: In a nutshell

"The basic idea behind ABC is that using a representative (enough) summary statistic η coupled with a small (enough) tolerance ϵ should produce a good (enough) approximation to the posterior..."

Marin et al. (2012)

Concluding remarks

- We considered examples related to Type Ia Supernova, Stellar IMF, Exoplanet Eccentricity
- Three main ABC decisions: summary statistic, distance function, tolerance
- Sequential ABC: need to specify a transition kernel and determine importance weights
- Approximate Bayesian Computation could be a useful tool in astronomy, **but** it should be used with care.

Concluding remarks

- We considered examples related to Type Ia Supernova, Stellar IMF, Exoplanet Eccentricity
- Three main ABC decisions: summary statistic, distance function, tolerance
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- Approximate Bayesian Computation could be a useful tool in astronomy, **but** it should be used with care.

THANK YOU!!!

Additional useful resources

- `http://approximatebayesiancomputational.wordpress.com/`
- Csilléry et al. (2010): Approximate Bayesian Computation (ABC) in practice
- Csillery et al. (2012): abc: an R package for approximate Bayesian computation (ABC)
- Jabot et al. (2013): EasyABC: performing efficient approximate Bayesian computation sampling schemes (R package)

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