Statistical Inference

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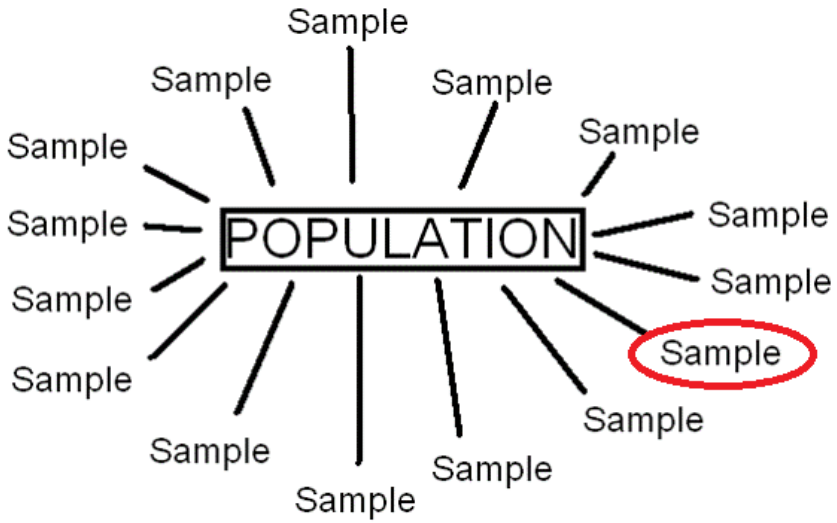
Overview of Statistical Inference



- Given a sample from a population, how do we make inferences about the population?
  - Estimation
    - Point estimation
    - Interval estimation
  - Hypothesis testing

Overview of Statistical Inference

- Some classical problems of statistical inference:
  - Tests and confidence intervals for an unknown population mean (one sample problem).
  - Tests and confidence intervals for the difference of two population means (two sample problem).
  - Tests for equality of several means (analysis of variance).
  - Tests for equality of several variances.
  - Chi-square goodness-of-fit test.

A Motivating Example

- Van den Bergh [1985] considered the luminosity function (LF) for globular clusters in various galaxies.
- V-d-B's conclusion: The LF for clusters in the Milky Way is adequately described by a normal distribution (with pdf:)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

- $M_0 \equiv \mu$: Mean visual absolute magnitude
- σ : Standard deviation of visual absolute magnitude

A Motivating Example

- Here is a diagram from Van den Bergh [1985], providing complete data for the M31 Globular Clusters in the Milky Way. (Notice that the data *appear* to be non-Gaussian)

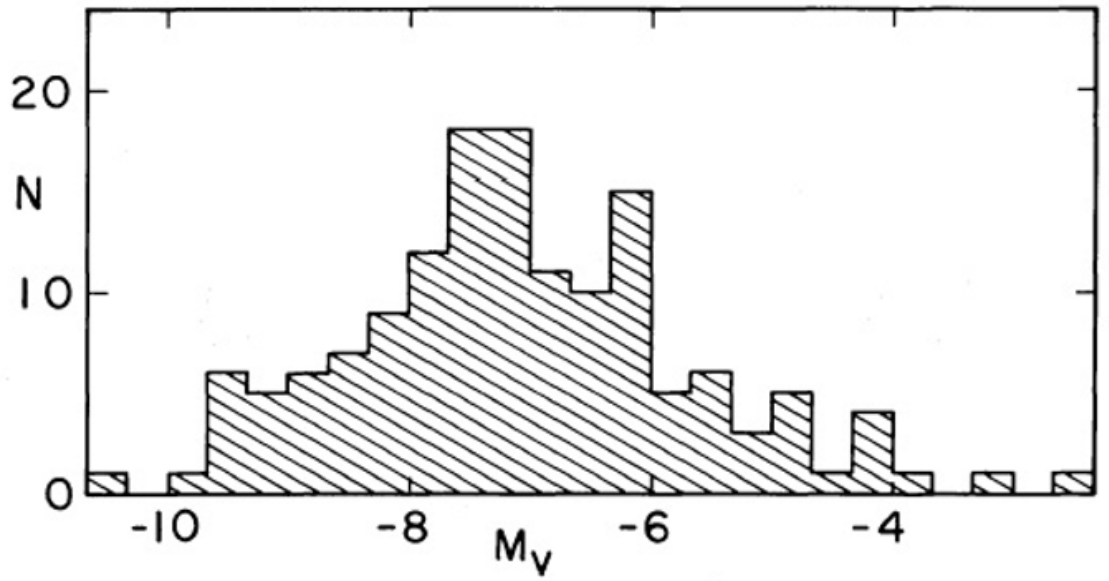


FIG. 2.—Luminosity function of Galactic globular clusters (one object at $M_V = -1.7$ is not plotted). Note that the luminosity function is asymetrical with a long tail extending to faint magnitudes.

A Motivating Example:

- From a statistical viewpoint, the main questions of interest are:
  1. On the basis of collected data, estimate the parameters μ and σ. Also, derive a plausible range of values for each parameter; etc.
  2. V-d-B, etc., conclude that the LF is "adequately described" by a normal distribution.
     How can we quantify the plausibility of their conclusion?

Some Definitions

- X: A random variable.
- *Population*: The collection of all values of X.
- $f(x;\boldsymbol{\theta})$: The prob. density function (p.d.f.) of X. For this lecture, we take $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$. Thus we have $\boldsymbol{\theta} = (\theta_1, \theta_2) = (\mu, \sigma)$ for the p.d.f. of the LF for Globular Clusters.
- *Statistical model*: A choice of p.d.f. for X. We choose a model which "adequately describes" data collected on X.
- *Parameter*: A number which describes a property of the population. $\theta_1, \ldots, \theta_k$ are parameters.
- *Parameter space*: The set of permissible values of the parameters. The parameter space for the p.d.f. of the LF example is

$$\Omega = \{(\mu, \sigma) : -\infty < \mu < \infty, \sigma > 0\}$$

Population Parameter vs Sample Statistic

- *Random sample*: Mutually independent random variables $X_1, \ldots, X_n$ which all have the same distribution as X

Difference between parameter and statistic

Parameter: A number computable only from the entire population
Statistic: A number computed from the random sample $X_1, \ldots, X_n$

- Some examples of statistics are

$$\text{the sample mean: } \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ , and}$$

$$\text{the sample variance: } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- In general, a *statistic* is a function $Y = t(X_1, \ldots, X_n)$ of the observations.

Sampling Distributions

- The sampling distribution for a statistic is the probability distribution of possible values of the statistic for repeated samples of the same size taken from the same population.
- For example, recall:

Standardized z- and t-Statistics for $\bar{x}$.

If a random sample is taken from a normal population then the standardized statistic,

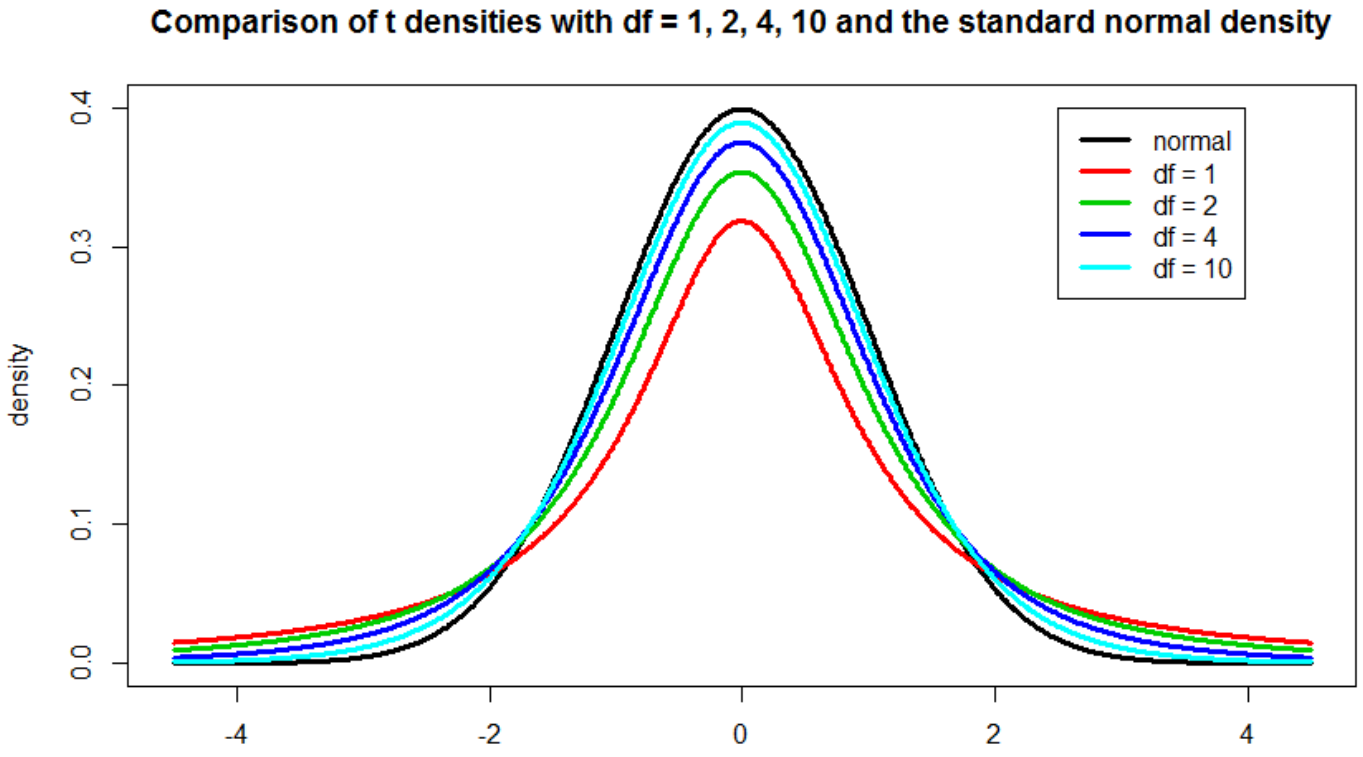
$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

has a standard normal distribution, and

$$\frac{\bar{X} - \mu}{S\sqrt{n}}$$

has a t-distribution with degrees of freedom equal to $n - 1$.

The t distribution



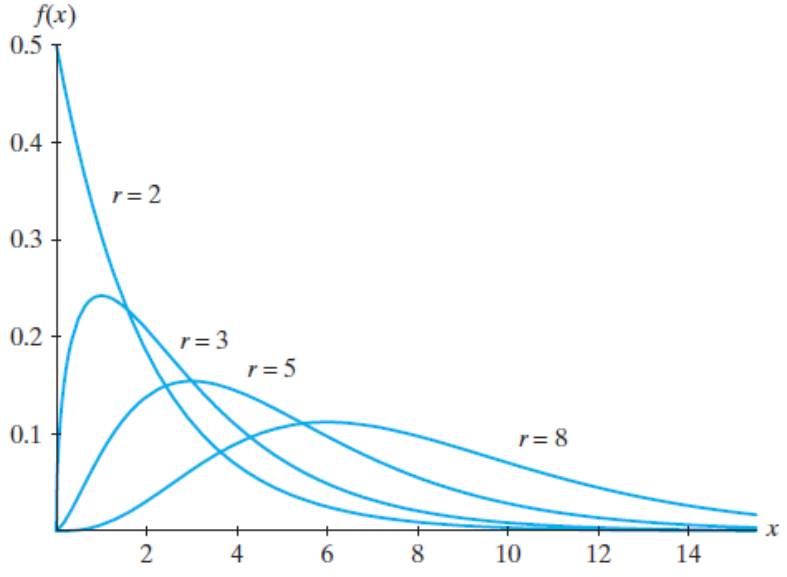
Sampling Distributions

- Another important result for normal random samples:

Distribution of S^2

Let $X_1, X_2, \ldots, X_n$ be observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$. Then the sample mean, $\bar{X}$ and the sample variance, S^2 are independent and

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$



Chi-square pdfs with $r = 2, 3, 5, 8$

Estimation

- Consider a random variable X with density $f(x, \theta)$.
- Let θ be an unknown parameter.
- Once θ is known, the density function is completely specified.

The problem of estimation

Given a random sample $x_1, \ldots, x_n$ from a population with above density, how to estimate the value of the unknown parameter θ.

- A statistic $t(X_1, \ldots, X_n)$ whose value is used to represent or estimate the unknown parameter θ is called a point estimator or estimator for short.
- $\hat{\theta}$ will represent an estimator.
- We distinguish between an estimator (a random variable) and an estimate (a specific value of the estimator).

Return to LF Example

- X: LF for globular clusters
- Model: $N(\mu, \sigma^2)$, the normal distribution with mean μ and variance σ^2
- Problem: Given a random sample $x_1, \ldots, x_n$, estimate μ
- $\bar{x}$ is a very good estimate of μ
- m, the sample median, is a good plausible estimate of μ
- $x_{(n)}$, the largest observed value in the LF, is obviously a poor estimate of μ, *since it almost certainly is much larger than* μ.
- Statistics like $\bar{X}, M$ and $X_{(n)}$ are examples of *point estimators* of μ.

Point Estimation

- We focus on two concepts:
  1. Methods for constructing an estimator
     - The method of moments
     - The method of maximum likelihood
     - Bayesian methods
     - Decision-theoretic methods
  2. Evaluating and choosing estimators from a class

The Method of Moments

- X: Random variable with p.d.f. $f(x; \theta_1, \theta_2)$
- Parameters to be estimated: θ_1, θ_2
- Random sample: $X_1, \ldots, X_n$

1. Calculate the first two sample moments:

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i, \;\; m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

2. Calculate $E(X)$ and $E(X^2)$, the first two population moments:

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x; \theta_1, \theta_2) dx$$

   The results are in terms of θ_1 and θ_2

3. Solve for θ_1, θ_2 in the simultaneous equations:

$$E(X) = m_1, \;\; E(X^2) = m_2$$

   The solutions are the *method-of-moments estimators* of θ_1, θ_2

Method of Moments Example: LF for Globular Clusters

- We are given a random sample: $X_1, \ldots, X_n$ from $N(\mu, \sigma^2)$
  1. Obtain the first two sample moments:

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{n-1}{n}S^2 + \bar{X}^2$$

  2. Obtain the first two population moments:

$$E(X) = \int_{-\infty}^{\infty} x f(x; \mu, \sigma^2) dx = \mu$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x; \mu, \sigma^2) dx = \mu^2 + \sigma^2$$

  3. Solve: $\hat{\mu} = m_1, \hat{\mu}^2 + \hat{\sigma}^2 = m_2$
     Solution: $\hat{\mu} = \bar{X}, \hat{\sigma}^2 = m_2 - m_1^2 = \frac{n-1}{n}S^2$

The Method of Maximum Likelihood

- R. A. Fisher (1912), "On an absolute criterion for fitting frequency curves," Messenger of Math. 41, 155160
- Fisher's first mathematical paper, written while a final-year undergraduate in mathematics and mathematical physics at Cambridge University
- It's not clear what motivated Fisher to study this subject; perhaps it was the influence of his tutor, the astronomer F. J. M. Stratton.
- Fisher's paper started with a criticism of two methods of curve fitting, least-squares and the method of moments.

Maximum Likelihood Estimation

- Recall: LF for globular clusters in the Milky Way; van den Bergh's normal model,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

- μ: Mean visual absolute magnitude
- σ: Std. deviation of visual absolute magnitude
- $\bar{X}$ is a good estimator for μ; S^2 is a good estimator for σ^2. (More on this later!)
- We seek a method which produces good estimators automatically
- Fisher's brilliant idea: The method of maximum likelihood

Maximum Likelihood Estimation

- Choose a globular cluster at random; what is the chance that the LF will be exactly -7.1 mag? Exactly -7.2 mag?
- For any continuous random variable X,

$$P(X = c) = 0$$

- Suppose $X \sim N(\mu = -6.9, \sigma^2 = 1.21)$
- X has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

- $\mathrm{P}(X = -7.1) = 0$, but

$$\begin{aligned} f(-7.1) &= \frac{1}{1.1\sqrt{2\pi}} \exp\left[-\frac{(-7.1+6.9)^2}{2(1.1)^2}\right] \\ &= 0.37 \end{aligned}$$

Maximum Likelihood Estimation

- Interpretation: In one simulation of the random variable X, the "likelihood" of observing the number -7.1 is 0.37
- $f(-7.2) = 0.28$
- In one simulation of X, the value $x = -7.1$ is 32% more likely to be observed than the value $x = -7.2$
- $x = -6.9$ is the value which has the greatest (or maximum) likelihood, for it is where the probability density function is at its maximum

Maximum Likelihood Estimation

- Return to a general model $f(x; \boldsymbol{\theta})$
- Random sample: $X_1, \ldots, X_n$
- Recall that the X_i are independent random variables
- The *joint* probability density function of the sample is

$$f(x_1; \boldsymbol{\theta}) f(x_2; \boldsymbol{\theta}) \cdots f(x_n; \boldsymbol{\theta})$$

- Here the variables are the X's, while $\boldsymbol{\theta}$ is fixed
- Fishers brilliant idea: Reverse the roles of the X's and $\boldsymbol{\theta}$
- Regard the X's as fixed and $\boldsymbol{\theta}$ as the variable

Maximum Likelihood Estimation

- The *likelihood function* is

$$L(\boldsymbol{\theta}; X_1, \ldots, X_n) = f(x_1; \boldsymbol{\theta}) f(x_2; \boldsymbol{\theta}) \cdots f(x_n; \boldsymbol{\theta})$$

- Simpler notation: $L(\boldsymbol{\theta})$
- $\hat{\boldsymbol{\theta}}$, the maximum likelihood estimator of $\boldsymbol{\theta}$, is the value of $\boldsymbol{\theta}$ where L is maximized
- $\hat{\boldsymbol{\theta}}$ is a function of the X's
- Usually, it is easier to find the maximum of the logarithm of the likelihood.
- Caution: The MLE is not always unique.

Obtaining MLEs: Example 1

- Example: " . . . cosmic ray composition - The path length distribution . . . "
- X: Length of paths
- Model: The exponential distribution,

$$f(x;\theta) = \theta^{-1} exp\left(-x/\theta\right), \; x > 0$$

- Parameter: $\theta > 0$
- Given random sample: $X_1, \ldots, X_n$
- Likelihood function:

$$\begin{aligned} L(\theta) &= f(X_1;\theta) f(X_2;\theta) \cdots f(X_n;\theta) \\ &= \theta^{-n} exp\left(-\left(X_1 + \cdots + X_n\right)/\theta\right) \\ &= \theta^{-n} exp\left(-n\bar{X}/\theta\right) \end{aligned}$$

Obtaining MLEs: Example 1

- To maximize L, we use calculus
- It is also equivalent to maximize ln L:

$$\ln L(\theta) = -n \ln (\theta) - n\bar{X}\theta^{-1}$$

$$\frac{d}{d\theta} \ln L(\theta) = -n\theta^{-1} + n\bar{X}\theta^{-2}$$

$$\frac{d^2}{d\theta^2} \ln L(\theta) = n\theta^{-2} - 2n\bar{X}\theta^{-3}$$

- Solve the equation $\frac{d}{d\theta} \ln L(\theta) = 0$

$$\theta = \bar{X}$$

- Check that $\frac{d^2}{d\theta^2} \ln L(\theta) < 0$ at $\theta = \bar{X}$
- $\ln L(\theta)$ is maximized at $\theta = \bar{X}$
- Conclusion: The MLE of θ is $\hat{\theta} = \bar{X}$

Obtaining MLEs: Example 2a

- LF for globular clusters; $X \sim N(\mu, \sigma^2)$

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

- Assume that σ is known (1.1 mag, say)
- Random sample: $X_1, \ldots, X_n$
- Likelihood function:

$$\begin{aligned} L(\mu) &= f(X_1; \mu) f(X_2; \mu) \cdots f(X_n; \mu) \\ &= (2\pi)^{-n/2} \sigma^{-n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2\right] \end{aligned}$$

- Maximize ln L using calculus: $\hat{\mu} = \bar{X}$

Obtaining MLEs: Example 2b

- LF for globular clusters; $X \sim N(\mu, \sigma^2)$
- Both μ and σ are unknown
- A likelihood function of two variables,

$$L(\mu, \sigma^2) = f(X_1; \mu, \sigma^2) \cdots f(X_n; \mu, \sigma^2)$$
$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2\right]$$

$$\ln L = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$$

$$\frac{\partial}{\partial\mu}\ln L = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)$$

$$\frac{\partial}{\partial(\sigma^2)}\ln L = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(X_i - \mu)^2$$

Obtaining MLEs: Example 2b

- Solve for μ and σ^2 in the simultaneous equations:

$$\frac{\partial}{\partial \mu} \ln L = 0, \quad \frac{\partial}{\partial(\sigma^2)} \ln L = 0$$

- We also verify that L is concave at the solutions of these equations (Hessian matrix)
- Conclusion: The MLEs are

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

- $\hat{\mu}$ is unbiased: $E(\hat{\mu}) = \mu$
- $\hat{\sigma}^2$ is not *unbiased*: $E(\hat{\sigma}^2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$
- For this reason, we use $\frac{n-1}{n}\hat{\sigma}^2 \equiv S^2$

Obtaining MLEs: Example 3

- Calculus cannot always be used to find MLEs
- Model:

$$f(x;\theta) = \begin{cases} \exp\ [-(x-\theta)]\,, & x \geq \theta \\ 0, & x < \theta \end{cases}$$

- Parameter: $\theta > 0$
- Given random sample: $X_1, \ldots, X_n$

$$L(\theta) = f(X_1;\theta) \cdots f(X_n;\theta)$$

$$= \begin{cases} \exp\ \left(-\sum_{i=1}^{n}(X_i-\theta)\right), & \text{all } X_i \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

- $X_{(1)}$: The smallest observation in the sample
- "all $X_i \geq \theta$" is equivalent to "$X_{(1)} \geq \theta$"

$$L(\theta) = \begin{cases} \exp\ \left(-n(\bar{X}-\theta)\right), & \theta \leq X_{(1)} \\ 0, & \text{otherwise} \end{cases}$$

- Conclusion: $\hat{\theta} = X_{(1)}$

Numerical Methods for ML Estimation

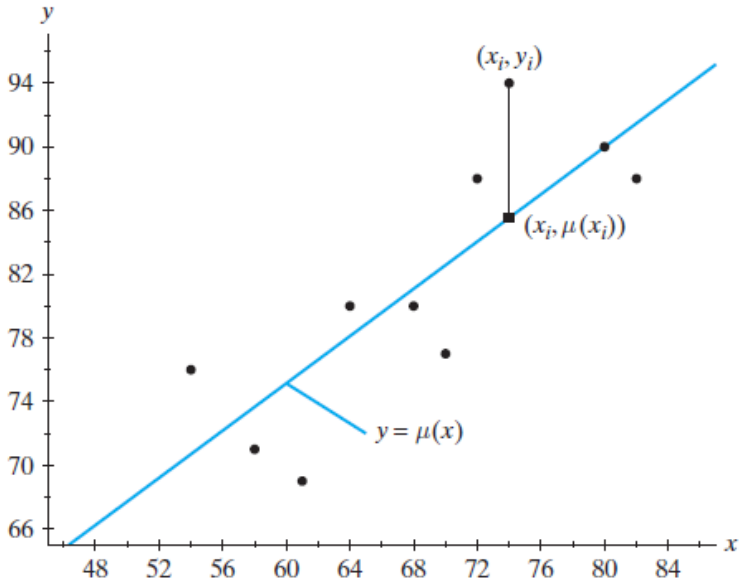
- In general, solutions to the likelihood equation cannot be obtained in closed form.
- Algorithms for optimization and root finding that can be employed.
  - Newton-Raphson Algorithm
  - EM (Expectation/Maximization) Algorithm

The ML Method for Linear Regression Analysis

- Scatterplot data: $(x_1, y_1), \ldots, (x_n, y_n)$
- Basic assumption: The x_is are non-random measurements; the y_i are observations on Y, a random variable
- Statistical model:

$$Y_i = \alpha + \beta x_i + \epsilon_i, \;\; i = 1, \ldots, n$$

- Errors $\epsilon_1, \ldots, \epsilon_n$: a random sample from $N(0, \sigma^2)$



- Parameters: α, β, σ^2
- $Y_i \sim N(\alpha + \beta x_i, \sigma^2)$: The Y_i are independent
- The Y_i are not identically distributed, because they have differing means

The ML Method for Linear Regression Analysis

- The likelihood function is the joint density function of the observed data, $Y_1, \ldots, Y_n$

$$L(\alpha, \beta, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left[-\frac{\sum_{i=1}^{n}(Y_i - \alpha - \beta x_i)^2}{2\sigma^2}\right]$$

- Maximize $\ln L$ over all α, β and $\sigma^2 > 0$.

The ML estimators are:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \ \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x} \text{ and}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\alpha} - \hat{\beta}x_i)^2$$

Invariance Property of the MLE

Theorem

Let $\hat{\theta}$ be the MLE of θ in the density $f(x;\theta)$ where θ is assumed to be one-dimensional. If h is a one-to-one function, then the MLE of $h(\theta)$ is $h(\hat{\theta})$.

- Example: Consider the normal density with $\mu = \mu_0$ known. It can be shown that the MLE of σ^2 is

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^2.$$

By the invariance property, the MLE of σ is

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^2}.$$

You Can Usually Bet on the MLE!

- The method of maximum likelihood works well when intuition fails and no obvious estimator can be found.
- When an obvious estimator exists the method of ML often will find it.
- The method can be applied to many statistical problems: regression analysis, analysis of variance, discriminant analysis, hypothesis testing, principal components, etc.
- We will return to describe some further properties of the MLE after we have developed some language for comparing estimators.

Bias of an Estimator

- Let $\hat{\theta}$ be an estimator for a parameter θ. We define the **bias** of the estimator as:

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$$

- If we use the sample mean $\bar{X}$ to estimate the population mean μ, then the bias of $\bar{X}$ is:

$$Bias(\bar{X}) = E(\bar{X}) - \mu$$

  Similarly, if we use the sample variance S^2 to estimate the population variance σ^2, then the bias of S^2 is:

$$Bias(S^2) = E(S^2) - \sigma^2$$

- Whenever the bias of an estimator is equal to 0, we call the estimator an unbiased estimator for the parameter of interest.
- It is clear that the sample mean is *unbiased* for the population mean. It can be shown that the sample variance is also unbiased for the population variance.

Bias of an Estimator

- Example: The Luminosity Function LF for globular clusters
- The sample mean, $\bar{X}$, is unbiased:

$$E\left(\bar{X}\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu$$

- The largest observed LF $X_{(n)}$ is not unbiased: $E(X_{(n)}) > \mu$

Variance and Standard Error of an Estimator

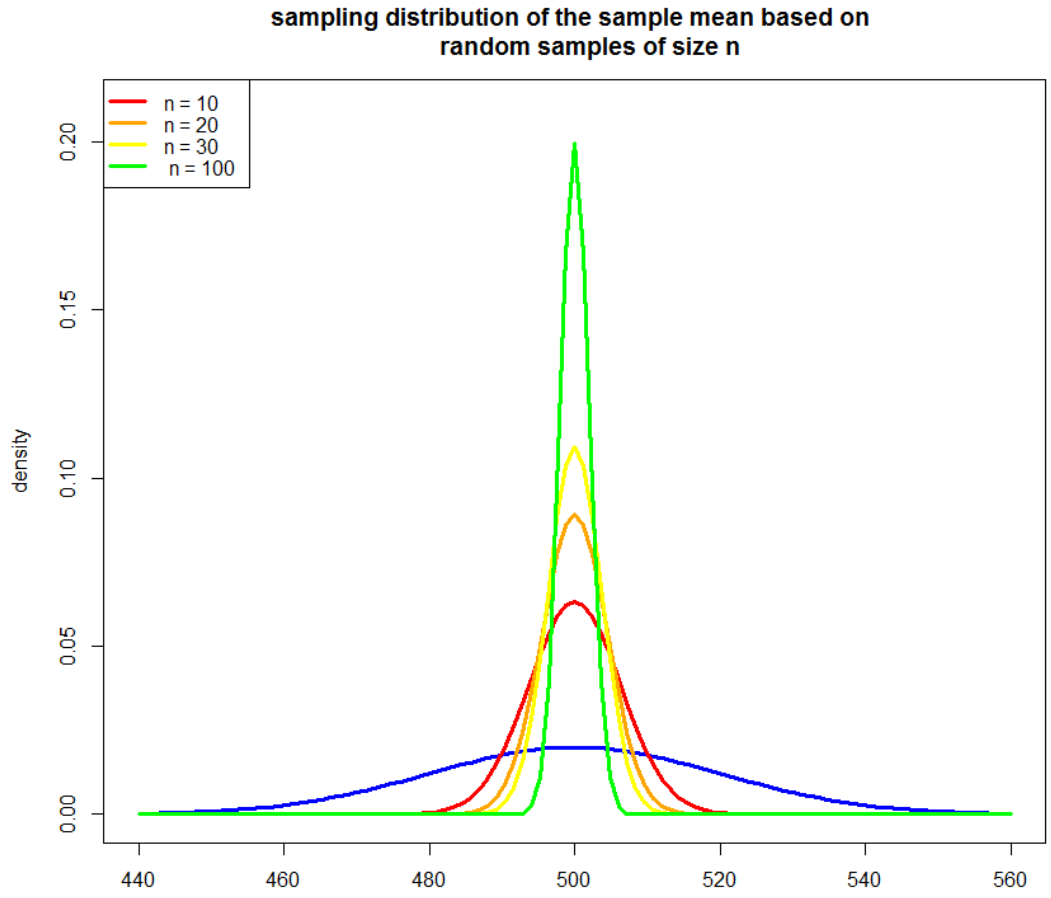
- Recall that an estimator is a random variable. When the variance of $\hat{\theta}$ exists, its standard deviation (also called *standard error*) is given by $\sigma_{\hat{\theta}} = \sqrt{\text{Var}(\hat{\theta})}$.
- As an example, suppose that we have a random sample from a distribution with mean μ and finite variance σ^2. The distribution of $\bar{X}$ has mean μ and variance $\frac{\sigma^2}{n}$.
- Thus we say that the standard error of the sample mean $\bar{X}$ is:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

- Since σ is usually unknown, we can estimate it with the sample standard deviation, S. In that case, we obtain the estimated standard error of the sample mean $\bar{X}$ as:

$$\hat{\sigma}_{\bar{X}} = \frac{S}{\sqrt{n}}$$

Effect of Sample Size on Sampling Distribution



The blue curve is the density of the original population from which samples are taken.

MSE of an Estimator

- We also want estimators which are "close" to θ
- For any estimator $\hat{\theta}$, we define the mean square error (MSE) as

$$E\left(\hat{\theta}-\theta\right)^2.$$

- Choose as our point estimator the estimator for which the MSE is smallest
- It can be shown that:

$$MSE(\hat{\theta}) = \text{Var}(\hat{\theta}) + (\text{bias}(\hat{\theta}))^2$$

- An estimator $\hat{\theta}$ which minimizes MSE is said to have *minimum mean square error*
- If $\hat{\theta}$ is also unbiased then $MSE = \text{Var}\,(\hat{\theta})$, and $\hat{\theta}$ is a *minimum variance unbiased estimator (MVUE)*

Comparing Estimators: An Example

Reminder

If R_1, R_2 are random variables and a, b are constants then

$$E(aR_1 + bR_2) = aE(R_1) + bE(R_2) :$$

If R_1 and R_2 are also independent then

$$Var(aR_1 + bR_2) = a^2 Var(R_1) + b^2 Var(R_2) :$$

- Random sample of size $n = 3 : X_1, X_2, X_3$
- Two point estimators of μ
- Sample mean: $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$ versus
- A weighted average: $Y = \frac{1}{6}(X_1 + 2X_2 + 3X_3)$
- Both estimators are unbiased: $E(\bar{X}) = \mu$, and

$$\begin{aligned} E(Y) &= \frac{1}{6}E(X_1 + 2X_2 + 3X_3) \\ &= \frac{1}{6}(\mu + 2\mu + 3\mu) = \mu \end{aligned}$$

Comparing Estimators: An Example

- However,

$$\text{Var}\,(\bar{X}) = \frac{1}{3^2}(\sigma^2 + \sigma^2 + \sigma^2) = \frac{1}{3}\sigma^2$$

- while

$$\begin{aligned}\text{Var}\,(Y) &= \frac{1}{6^2}\ \text{Var}\,(X_1 + 2X_2 + 3X_3) \\ &= \frac{1}{36}(\sigma^2 + 2^2\sigma^2 + 3^2\sigma^2) = \frac{7}{18}\sigma^2\end{aligned}$$

- $\bar{X}$ and Y are unbiased but $\text{Var}\,(\bar{X}) < \text{Var}\,(Y)$
- The distribution of $\bar{X}$ is more concentrated around μ than the distribution of Y
- $\bar{X}$ is a better estimator than Y
- Note: For any sample size n, $\text{Var}\,(\bar{X}) = \frac{\sigma^2}{n}$

Comparing Estimators: Another Example

- Suppose we want to estimate a population proportion, p. We obtain X "successes" in n trials.
- The usual (and natural) estimator of p is the sample proportion $\hat{p} = \frac{X}{n}$. $\hat{p}$ is also the MLE and the MOM estimator of p.
- It is unbiased, and has $\text{MSE}(\hat{p}) = \text{Var}(\hat{p}) = \frac{p(1-p)}{n}$
- An alternative estimator is

$$\tilde{p} = \frac{X+2}{n+4}$$

- This estimator is the sample proportion of successes computed after adding two successes and two failures to the sample.
- Can you think of an intuitive justification for this estimator?

Comparing Estimators: Another Example

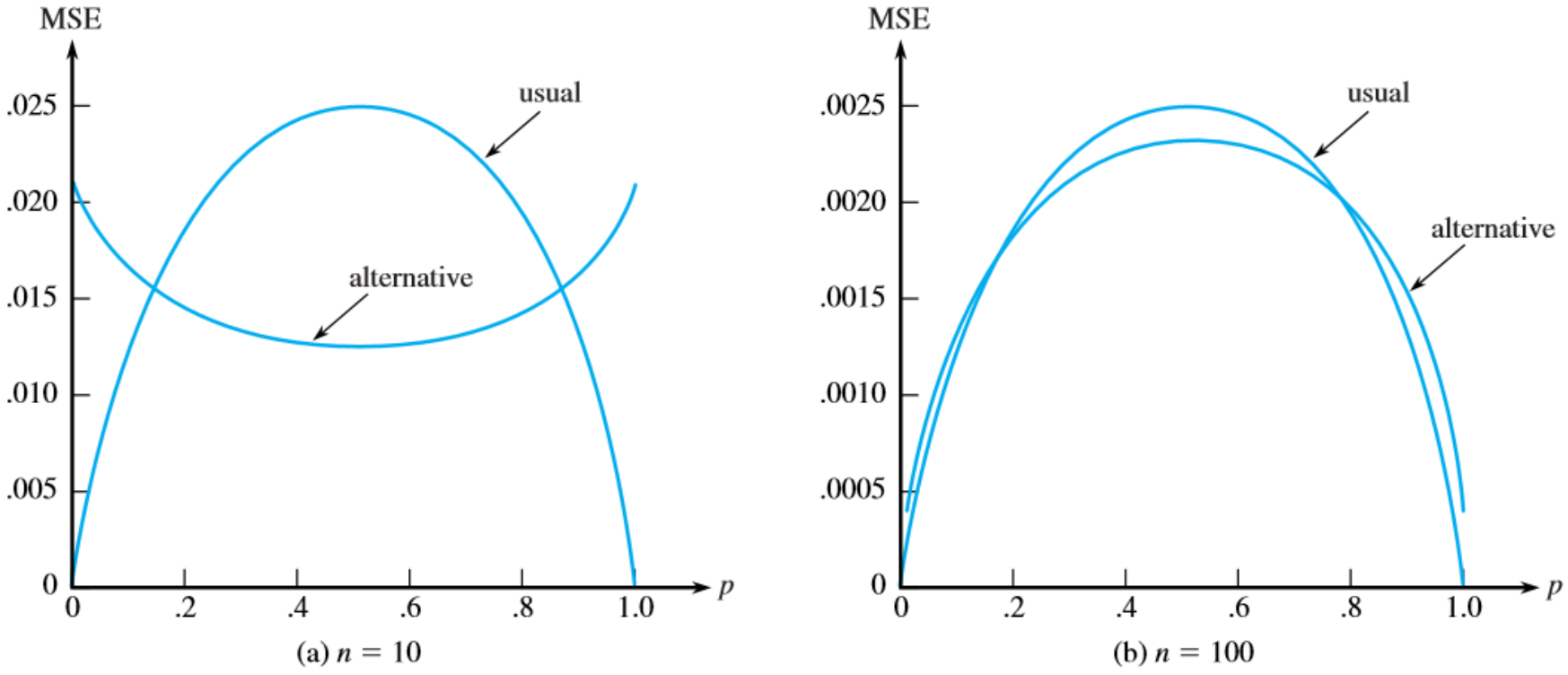


Figure 1: Comparison of MSE for the usual and alternative estimators of p. Taken from Devore and Berk [2007]

Loss and Risk

- Besides the MSE, there are other ways of quantifying closeness to the true parameter value. A general framework is to use loss functions.
- A *loss function* is a real-valued function of two variables, $L(\theta, a)$, where $\theta \in \Omega$ and a is a a real number. We interpret this to mean that the statistician loses $L(\theta, a)$ if the parameter equals θ and the estimate equals a.
- Some examples of loss functions are:
  - Squared error loss

$$L(\theta, a) = (\theta - a)^2$$

  - Absolute error loss

$$L(\theta, a) = |\theta - a|$$

Loss and Risk

- For a given loss function $L(\theta, a)$, the risk function of an estimator $\hat{\theta}$ is defined to be

$$R_{\hat{\theta}}(\theta) = E_{\theta}\left[L(\theta, a)\right]$$

- The risk function is an average loss.
- Consider the loss functions listed above. The corresponding risks are given by:

$$R_{\hat{\theta}}(\theta) = E(\hat{\theta} - \theta)^2$$

and

$$R_{\hat{\theta}}(\theta) = E\left|\hat{\theta} - \theta\right|$$

- Note that the first is the familiar mean squared error. The second is called the mean absolute error.

Consistency of an Estimator

- Random sample: $X_1, \ldots, X_n$
- $Y = t(X_1, \ldots, X_n)$: An estimator of θ
- Bear in mind that Y depends on n
- It would be good if Y "converges" to θ as $n \to \infty$
- Y is *consistent* if, for any $t > 0$,

$$P\left(|Y - \theta| \geq t\right) \to 0$$

as $n \to \infty$

- The Law of Large Numbers: If $X_1, \ldots, X_n$ is a random sample from X then for any $t > 0$,

$$P\left(|\bar{X} - \mu| \geq t\right) \to 0$$

as $n \to \infty$

- Very Important Conclusion: For any population, $\bar{X}$ is a consistent estimator of μ.

The Cramér-Rao or Information Inequality

- Given two unbiased estimators, we prefer the one with smaller variance
- In our quest for unbiased estimators with minimum possible variance, we need to know how small their variances can be
- X: Random variable with model $f(x;\theta)$
- The "support" of f is the region where $f > 0$.
- We assume that the "support" of f does not depend on θ.
- Given: a random sample: $X_1, \ldots, X_n$. Let Y be an unbiased estimator of θ.
- The Cramér-Rao Inequality: The smallest possible value that $Var(Y)$ can attain is $1/B$ where

$$B = nE\left[\frac{\partial}{\partial\theta}\ln\, f(X;\theta)\right]^2 = -nE\left[\frac{\partial^2}{\partial\theta^2}\ln\, f(X;\theta)\right]$$

- Example: " . . . cosmic ray composition - The path length distribution . . . "
- X: Length of paths
- Parameter: $\theta > 0$
- Model: $f(x;\theta) = \theta^{-1}\exp(-x/\theta), \quad x > 0$

$$\ln f(X;\theta) = -\ln \theta - \theta^{-1}X$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta) = \theta^{-2} - 2\theta^{-3}X$$

$$\begin{aligned} E\left[\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right] &= E\left(\theta^{-2} - 2\theta^{-3}X\right) \\ &= \theta^{-2} - 2\theta^{-3}E(X) \\ &= \theta^{-2} - 2\theta^{-3}\theta \\ &= -\theta^{-2} \end{aligned}$$

- The smallest possible value of $Var(Y)$ is θ^2/n
- This is attained by $\bar{X}$. For this problem, $\bar{X}$ is *the* best unbiased estimator of θ

Efficiency of Estimators

- Y : An unbiased estimator of a parameter θ
- We compare $\text{Var}(Y)$ with $1/B$, the lower bound in the Cramér-Rao inequality:
- $\frac{1}{B} \div \text{Var}(Y)$
- This number is called the *efficiency* of Y
- Obviously, $0 \leq$ efficiency ≤ 1
- If Y has 50% efficiency then about $1/0.5 = 2$ times as many sample observations are needed for Y to perform as well as the MVUE.
- The use of Y result in confidence intervals which generally are longer than those arising from the MVUE.
- If the MLE is unbiased then as n becomes large, its efficiency increases to 1.

General Properties of the MLE $\hat{\theta}$

We summarize here some properties of the MLE using the language we have learned in the last few slides:

1. $\hat{\theta}$ may not be unbiased. We often can remove this bias by multiplying $\hat{\theta}$ by a constant.
2. For many models, $\hat{\theta}$ is consistent.
3. The Invariance Property: For many nice functions g, if $\hat{\theta}$ is the MLE of θ then $g(\hat{\theta})$ is the MLE of $g(\theta)$.
4. The Asymptotic Property: For large n, $\hat{\theta}$ has an approximate normal distribution with mean θ and variance $1/B$ where

$$B = nE\left[\frac{\partial}{\partial\theta}\ln f(X;\theta)\right]^2$$

The asymptotic property can be used to develop confidence intervals for θ.

Confidence Intervals

- Recall: LF for globular clusters in the Galaxy
- X is $N(\mu, \sigma^2)$
- Given a random sample: $X_1, \ldots, X_n$, the sample mean $\bar{X}$ is an unbiased estimator of μ : $E(\bar{X}) = \mu$.
- What is the probability distribution of $\bar{X}$?
- $\bar{X}$ has a normal distribution

$$E(\bar{X}) = \mu, \text{ Var } (\bar{X}) = \frac{\sigma^2}{n}, \text{ so } \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

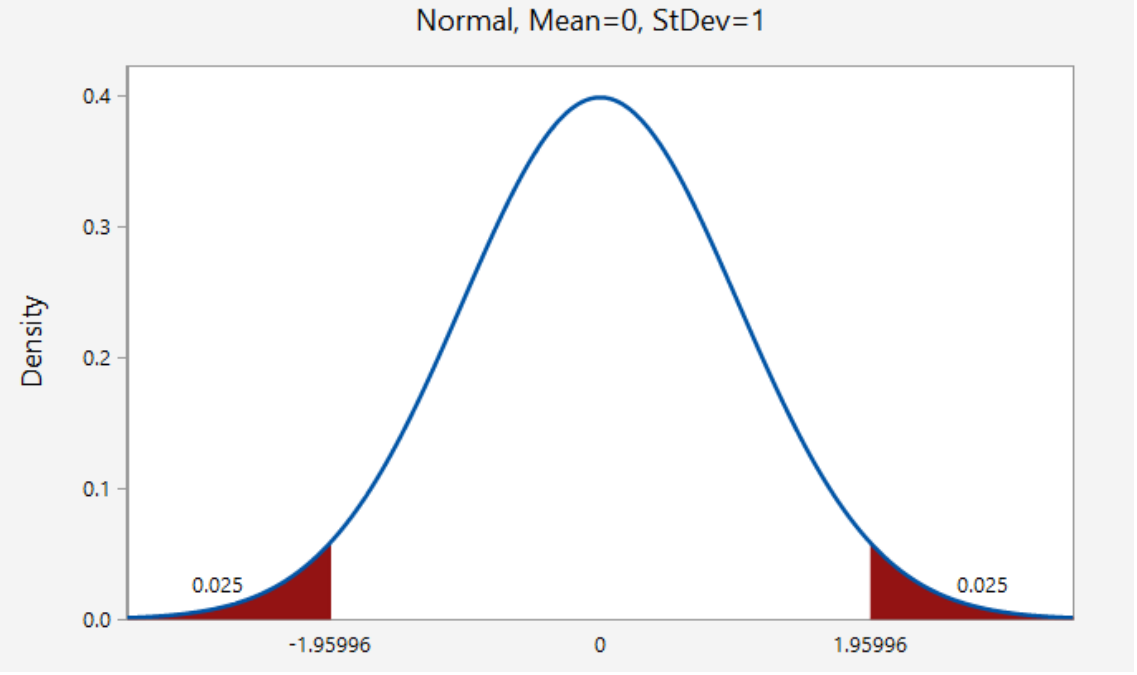
$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

- We want to use this fact to obtain an interval estimate for μ.

Confidence Intervals

- Consult the tables of the $N(0,1)$ distribution:

$$P(-1.96 < Z < 1.96) = 0.95$$



- For LF data,

$$P(-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96) = 0.95$$

Confidence Intervals

- Assume that σ is known, $\sigma = 1.2$ mag for Galactic globulars (Van den Bergh [1985])
- Solve for μ in the inequalities

$$-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96$$

- The solution is

$$\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$$

$$\Longrightarrow P\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

- The probability that the interval

$$\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

  "captures" μ is 0.95.
- This interval is called a 95% *confidence interval for* μ
- It is a plausible range of values for μ together with a quantifiable measure of its plausibility
- Notes:
  - A confidence interval is a *random* interval; it changes as the collected data changes. This explains why we say "**a** 95% confidence interval" rather than "the 95% confidence interval"
  - We chose the "cutoff limits" ± 1.96 symmetrically around 0 to minimize the length of the confidence interval.

Example (Devised from Van den Bergh [1985]):

- $n = 148$ Galactic globular clusters
- $\bar{x}$ = -7.1 mag
- We assume that $\sigma = 1.2$ mag
- $M_0 \equiv \mu$: The population mean visual absolute magnitude
- A 95% confidence interval for M_0 is

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$
$$= \left(-7.1 - 1.96\frac{1.2}{\sqrt{148}}, -7.1 + 1.96\frac{1.2}{\sqrt{148}}\right)$$
$$= (-7.1 \pm 0.193)$$

- This is a plausible range of values for M_0.

- The Warning: Don't bet your life that your 95% confidence interval has captured μ!
- Intervals with higher levels of confidence, 90%, 98%, 99%, 99.9%, can be obtained similarly.
- Intervals with confidence levels $100(1-\alpha)\%$ are obtained by replacing the multiplier 1.96 in a 95% confidence by $z_{\alpha/2}$, where $z_{\alpha/2}$ is determined by

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha;$$

  a 95% confidence has $\alpha = 0.05$.
- 90%, 98%, 99%, 99.9% confidence intervals correspond to $\alpha =$ .10,.02,.01, and .001, respectively; the corresponding values of $z_{\alpha/2}$ are 1.645,2.33,2.58, and 3.09, respectively.

Confidence Interval for μ with σ Unknown

- If σ is unknown then the previous confidence intervals are not useful
- A basic principle in statistics: Replace any unknown parameter with a good estimator
- LF data problem; a random sample $X_1, \ldots, X_n$ drawn from $N(\mu, \sigma^2)$
- We are tempted to construct confidence intervals for μ using the statistic $\frac{\bar{X}-\mu}{S/\sqrt{n}}$
- However, recall that it is not normally distributed, but has a t-distribution with $n-1$ degrees of freedom.

- We construct confidence intervals as before
- Suppose that $n = 16$, then see the tables of the t-distribution on 15 degrees of freedom:

$$P(-2.131 < T_{15} < 2.131) = 0.95$$

- Therefore

$$P\left(-2.131 < \frac{\bar{X} - \mu}{S/\sqrt{n}} < 2.131\right) = 0.95$$

- Solve for μ in the inequalities

$$-2.131 < \frac{\bar{X} - \mu}{S/\sqrt{n}} < 2.131$$

- A 95% confidence interval for μ is

$$\left(\bar{X} - 2.131\frac{S}{\sqrt{n}}, \bar{X} + 2.131\frac{S}{\sqrt{n}}\right)$$

- Example: $n = 16$, $\bar{x}$ = -7.1 mag, $s = 1.1$ mag.
- A 95% confidence interval for μ is -7.1 ± 0.586

Confidence Interval for σ^2

- We want to obtain confidence intervals for σ^2
- Random sample: $X_1, \ldots, X_n$ Normal population $N(\mu, \sigma^2)$
- S^2 is an unbiased and consistent estimator of σ^2
- Recall the sampling distribution of S^2:
- $\frac{(n-1)S^2}{\sigma^2}$ has a chi-squared distribution with $n-1$ degrees of freedom.
- We now construct confidence intervals as before by using the χ^2 distribution.

Confidence Interval for σ^2

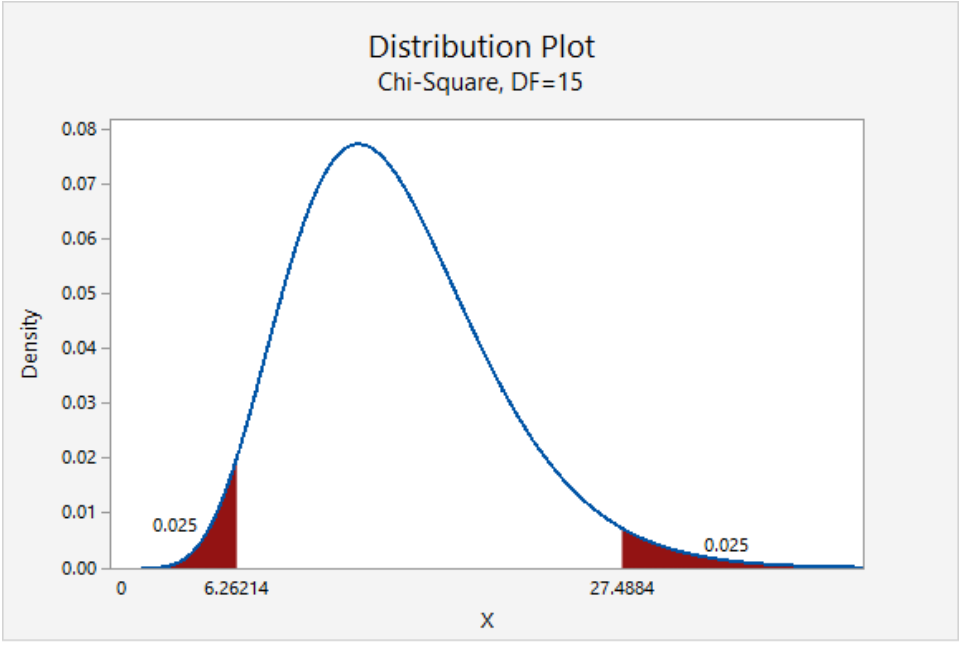
- Denote the percentage points by a and b

$$P\left(a < \chi^2_{n-1} < b\right) = 0.95$$

- We find a, b using tables of the χ^2 distribution
- Solve for σ^2 in the inequalities: $a < \frac{(n-1)S^2}{\sigma^2} < b$
- A 95% confidence interval for σ^2 is $\left(\frac{(n-1)S^2}{b}, \frac{(n-1)S^2}{a}\right)$

Confidence Interval for σ^2

- Example: $n = 16$, $s = 1.2$ mag
- Percentage points from the χ^2 tables (with 15 degrees of freedom): 6.262 and 27.488



- A 95% confidence interval for σ^2 is

$$\left(\frac{15 \times (1.2)^2}{27.488}, \frac{15 \times (1.2)^2}{6.262} \right) = (0.786, 3.449)$$

The Width of a Confidence Interval

The Width of a Confidence Interval

All other things remaining constant:
The greater the level of confidence, the longer the confidence interval.
The larger the sample size, the shorter the confidence interval.

- How do we choose n?
- In our 95% confidence intervals for μ, the term $1.96\sigma/\sqrt{n}$ is called the margin of error
- We choose n to have a desired margin of error
- To have a margin of error of 0.01 mag then we choose n so that

$$\frac{1.96\sigma}{\sqrt{n}} = 0.01$$

- Solve this equation for n:

$$n = \left(\frac{1.96\sigma}{0.01}\right)^2$$

Confidence Intervals with Large Sample Sizes

- Papers on LF for globular clusters
- Sample sizes are large: 68, 148, 300, 1000, ...
- A modified Central Limit Theorem
- $X_1, \ldots, X_n$: a random sample
- μ: The population mean
- $\bar{X}$ and S: The sample mean and std. deviation
- The modified CLT: If n is large then

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \approx N(0, 1)$$

- The conclusion does not depend on the population probability distribution
- The resulting confidence intervals for μ also do not depend on the population probability distribution

Tests of Hypotheses

- Alternatives to confidence intervals
- A LF researcher believes that $M_0 = -7.7$ mag for the M31 globular clusters. The researcher collects a *random sample* of data from M31
- A natural question: "Are the data strongly in support of the claim that $M_0 = -7.7$ mag?"
- *Statistical hypothesis*: A statement about the parameters of a population.
- *Statistical test of significance*: A procedure for comparing observed data with a hypothesis whose plausibility is to be assessed.
- *Null hypothesis*: The statement being tested.
- *Alternative hypothesis*: A competing statement.
- In general, the alternative hypothesis is chosen as the statement for which we hope to find supporting evidence.

Tests of Hypotheses

- In the case of our M31 LF researcher, the null hypothesis is $H_0 : M_0 = -7.7$
- An alternative hypothesis is $H_a : M_0 \neq -7.7$. This is a two-sided alternative hypotheses
- One-sided alternatives, e.g., $H_a : M_0 < -7.7$
- To test H_0 vs. H_a, we need:
  - (a) A test statistic: This statistic will be calculated from the observed data, and will measure the compatibility of H_0 with the observed data. It will have a sampling distribution free of unknown parameters.
  - (b) A rejection rule which specifies the values of the test statistic for which we reject H_0.

Tests of Hypotheses: An example

- Example: A random sample of 64 measurements has mean $\bar{x} = 5.2$ and std. dev. $s = 1.1$. Test the null hypothesis $H_0 : \mu = 4.9$ against the alternative hypothesis $H_a : \mu \neq 4.9$

  Step 1: The null and alternative hypotheses:
  $H_0 : \mu = 4.9, \ H_a : \mu \neq 4.9$

  Step 2: The test statistic:

$$T = \frac{\bar{X} - 4.9}{S/\sqrt{n}}$$

  Step 3: The distribution of the test statistic under the assumption that H_0 is valid: $T \approx N(0, 1)$

  Step 4: The rejection rule: Reject H_0 if $|T| > 1.96$. Otherwise, we fail to reject H_0. This choice of cutoff point (also called critical value) results in a 5% level of significance of the test of hypotheses.

Tests of Hypotheses: An example

Step 5: Calculate the value of the test statistic: The calculated value of the test statistic is

$$\frac{\bar{X} - 4.9}{S/\sqrt{n}} = \frac{5.2 - 4.9}{1.1/\sqrt{64}} = 2.18$$

Step 6: Decision: We reject H_0; the calculated value of the test statistic exceeds the critical value, 1.96.
We report that the data are *significant* and that there is a *statistically significant* difference between the population mean and the hypothesized value of 4.9

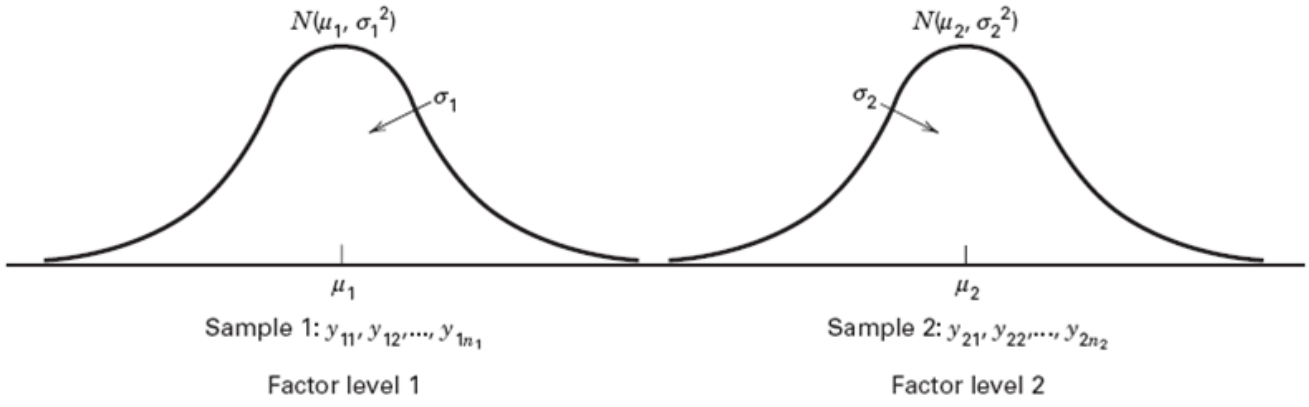
Step 7: The P-value of the test: The smallest significance level at which the data are significant.

$$P(t > 2.18) = 0.0165315$$

Thus the exact p-value is $2 \times 0.01653 = 0.03306$

Inference on the Means of Two Populations, Variances Known

- Premise: We have two different populations with unknown means μ_1 and μ_2, respectively and known variances σ_1^2 and σ_2^2, respectively. We have obtained two samples of sizes n_1 and n_2 from population 1 and population 2 respectively.



Inference on the Means of Two Populations, Variances Known

- Let us denote the samples as

$$\boldsymbol{X}_1 = X_{11}, X_{12}, ..., X_{1n_1}$$
$$\boldsymbol{X}_2 = X_{21}, X_{22}, ..., X_{2n_2}$$

- We make two further assumptions: The samples $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent; and both populations are normal, (or if not, then the conditions of the central limit theorem apply).
- At this point all we need to do are to establish procedures for making inference on the difference between the (unknown) means of the two populations, $\mu_1 = \mu_2$.
- The point estimator for the parameter of interest, $\mu_1 - \mu_2$ is $\bar{X}_1 - \bar{X}_2$.
- By standardizing we note that

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

has a standard normal distribution.

- Suppose someone makes a claim that the difference in means between populations 1 and 2 is Δ_0. (We call Δ_0 the null or hypothesized difference.)
- To test this claim, we formulate the problem as a hypothesis test with null hypothesis $\boxed{H_0 : \mu_1 - \mu_2 = \Delta_0}$.
- Then an appropriate test statistic is given by

$$Z_0 = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

- The remaining steps are similar to what we did in the case of a single mean.
- We usually have $\Delta_0 = 0$. In this case the null hypothesis $\boxed{H_0 : \mu_1 - \mu_2 = 0}$ OR $\boxed{H_0 : \mu_1 = \mu_2}$ can be put in words as: H_0: There is no difference in mean between population 1 and population 2.
- The two-tailed alternative is $\boxed{H_0 : \mu_1 - \mu_2 \neq 0}$ OR $\boxed{H_0 : \mu_1 \neq \mu_2}$. In words, there is a difference in mean between population 1 and population 2.
- We can also have a one-tailed alternative in either direction.

Inference on the Means of Two Populations, Variances Unknown - An Example

- Two random samples of sizes $n_1 = 15$ and $n_2 = 17$ are selected, and the sample means and sample variances are $\bar{x}_1 = 8.73, s_1^2 = 0.35, \bar{x}_2 = 8.68, \text{and} s_2^2 = 0.40$. Assume that $\sigma_1^2 = \sigma_2^2$ and that the data are drawn from a normal distribution.

  (a) Is there evidence to support the populations have different mean diameters? Use a P-value in arriving at the conclusion.

  (b) Construct a 95% CI for the difference in population means. Interpret this interval.

  Note that for problems of this nature, there are two possible scenarios which will lead to different distributions of the test statistics:

  1. We assume $\sigma_1^2 = \sigma_2^2$
  2. We assume $\sigma_1^2 \neq \sigma_2^2$.

(a) $\boxed{H_0 : \mu_1 - \mu_2 = 0, H_1 : \mu_1 - \mu_2 \neq 0}$

First we find the pooled sample variance, S_p^2.

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{14 \times 0.35 + 16 \times 0.40}{30}$$
$$= 0.377$$

Therefore $S_p = \sqrt{S_p^2} = 0.6137$.

Now compute the test statistic:

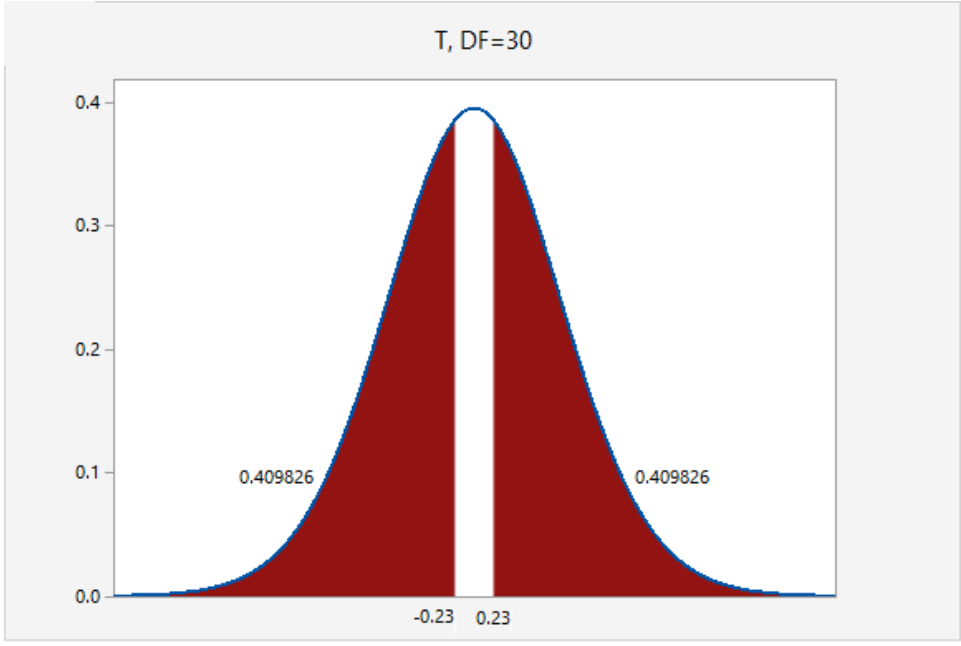
$$T = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$
$$= \frac{8.73 - 8.68 - 0}{0.6137\sqrt{\frac{1}{15} + \frac{1}{17}}}$$
$$= 0.230$$

$|t_0| = |0.230| = 0.230$

- The test statistic follows a t distribution with degrees of freedom $n_1 + n_2 - 2 = 30$. Therefore the P-value is

$$2 \times P(t_{30} > |t_0|) = 2 \times P(t_{30} > 0.230) > 0.80$$

We fail to reject H_0



- (b) From the t-table, $t_{\alpha/2,n_1+n_2-2} = t_{0.025,30} = 2.042$. The 95% CI is given by

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2,n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 \leq$$

$$\bar{X}_1 - \bar{X}_2 + t_{\alpha/2,n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$0.05 - 2.042(0.2174) \leq \mu_1 - \mu_2 \leq 0.05 + 2.042(0.2174)$$

$$-0.394 \leq \mu_1 - \mu_2 \leq 0.494$$

- Therefore, we can be 95% confident that the true mean difference lies between -0.394 and 0.494.

The Likelihood Ratio Test

- $X \sim N(\mu, \sigma^2)$; parameters μ and σ^2
- Model: $f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$
- Random sample: $X_1, \ldots, X_n$
- We wish to test $H_0 : \mu = 3$ vs. $H_a : \mu \neq 3$
- Parameter space: The space of all permissible values of the parameters

$$\Omega = \{(\mu, \sigma) : \infty < \mu < \infty, \sigma > 0\}$$

- H_0 and H_a represent restrictions on the parameters, so we are led to parameter subspaces

$$\omega_0 = \{(\mu, \sigma) : \mu = 3, \sigma > 0\}$$
$$\omega_a = \{(\mu, \sigma) : \mu \neq 3, \sigma > 0\}$$

The Likelihood Ratio Test

$$L(\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2\right]$$

- Maximize $L(\mu, \sigma^2)$ over ω_0 and ω_a
- The *likelihood ratio test statistic* is

$$\lambda = \frac{\max\limits_{\omega_0} L(\mu, \sigma^2)}{\max\limits_{\omega_0 \cup \omega_a} L(\mu, \sigma^2)} = \frac{\max\limits_{\sigma>0} L(3, \sigma^2)}{\max\limits_{\sigma>0, \mu} L(\mu, \sigma^2)}$$

- Fact: $0 \leq \lambda \leq 1$
- $L(3, \sigma^2)$ is maximized over ω_0 at

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - 3)^2$$

The Likelihood Ratio Test

$$\max_{\omega_0} L(3, \sigma^2) = L\left(3, \frac{1}{n}\sum_{i=1}^{n}(X_i - 3)^2\right)$$

$$= \left[\frac{n}{2\pi \exp \sum_{i=1}^{n}(X_i - 3)^2}\right]^{n/2}$$

$L(\mu, \sigma^2)$ is maximized over ω_a at

$$\mu = \bar{X},\ \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\max_{\omega_0 \cup \omega_a} L(\mu, \sigma^2) = L\left(\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right)$$

$$= \left[\frac{n}{2\pi \exp \sum_{i=1}^{n}(X_i - \bar{X})^2}\right]^{n/2}$$

The Likelihood Ratio Test

- The likelihood ratio test statistic:

$$\lambda = \left[\frac{n}{2\pi \exp \sum_{i=1}^{n}(X_i - 3)^2}\right]^{n/2} \div \left[\frac{n}{2\pi \exp \sum_{i=1}^{n}(X_i - \bar{X})^2}\right]^{n/2}$$

$$= \left[\sum_{i=1}^{n}(X_i - \bar{X})^2 \div \sum_{i=1}^{n}(X_i - 3)^2\right]^{n/2}$$

- λ is close to 1 iff $\bar{X}$ is close to 3
- λ is close to 0 iff $\bar{X}$ is far from 3
- This particular LRT statistic λ is equivalent to the t-statistic seen earlier
- In this case, the ML method discovers the obvious test statistic

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