Wavelet Variance Analysis of Irregularly Sampled Time Series

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Outline for the talk

- Brief **overview**.
- **New work** on **Slepian wavelet variances**

– Slepian **wavelet filters** on integer lattice.

– Slepian **wavelet variance** for **regularly sampled** data.

– extension to **irregularly sampled data**.

– Simulation study.

- Analysis of a **variable star** data
- Some **new ideas** based on Mondal and Percival (2008).

Wavelet variance for a time series

- Interest is on **second order properties**. This is **different** from curve estimation!

Time/space domain	Frequency domain
covariances, **variogram**	**Fourier** transform, **spectral** distribution

Wavelet domain

wavelet transform, wavelet variance and covariances

- Wavelet domain **complements** time/space and frequency domains!
- **Wavelets** decompose a stochastic process w.r.t. a set of **basis functions**.
- Each basis function is associated with a **particular scale**.
- Allow us to study **scale-based analysis** of a stochastic process.
- **Focus** on **wavelet domain** and **exploratory analysis**.

Wavelet variance for a time series

References for **theory**:

- Percival (1995), Percival and Walden (2000), Taqqu and coauthors, ...

Applications include

- Variation in **low-mass X-ray binaries** (Scargale et al., 1993).
- **Solar coronal** activity (Rybák & Dorotovič, 2002).
- Time deviation in **atomic clock**.
- Ocean **surface waves** (Massel, 2001).
- Relationship between **rainfall** and **runoff** (Labat et al., 2001).
- Variation in soil properties in **agricultural field** (Lark & Webster, 2001).
- **Heart rate** variability (Pichot et al., 1999).
- **DNA sequence** (Vannucci & Lio, 2001).

Wavelet variance for time series

Let $X_0, X_1, \ldots, X_{N-1}$ be the observed time series with dth order stationary increments.

Wavelet transform decomposes X_t by applying **Daubechies wavelet filters** $\{h_{jl}\}$:

$$W_{j,t} = \sum_{l=0}^{L_j-1} h_{jl} X_{t-l}; \qquad t = 0, \pm 1, \ldots; \qquad j = 1, 2, \ldots; \qquad L \geq 2d.$$

Wavelet coefficient $W_{j,t}$ is difference between weighted averages of 2^{j-1} time points.

Define **wavelet variance** at scale $\tau_j = 2^{j-1}$ as

$$\nu_X^2(\tau_j) = \text{var}\left(W_{j,t}\right). \tag{1}$$

Estimate $\nu_X^2(\tau_j)$ by

$$\hat{\nu}_X^2(\tau_j) \;=\; \frac{1}{N_j} \sum_{t=L_j-1}^{N-1} W_{j,t}^2, \qquad N_j \;=\; N - L_j + 1.$$

Use results of Percival (1995) to **construct confidence intervals**.

Rationale for wavelet variance

- For stationary process, wavelet variance **decomposes** $\mathrm{var}\{X_t\}$:

$$\mathrm{var}\{X_t\} = \sum_{j=1}^{\infty} \nu_X^2(\tau_j). \tag{2}$$

This is alternative to classical decomposition

$$\mathrm{var}\{X_t\} = \int_{-1/2}^{1/2} S_X(f)\, df.$$

$\nu_X^2(\tau_j)$ is contribution to $\mathrm{var}\{X_t\}$ due to scale $\tau_j = 2^{j-1}$.

- Provides a way of **regularizing** the spectrum:

$$\nu_X^2(\tau_j) \approx 2 \int_{2^{-(j+1)}}^{2^{-j}} S_X(f)\, df.$$

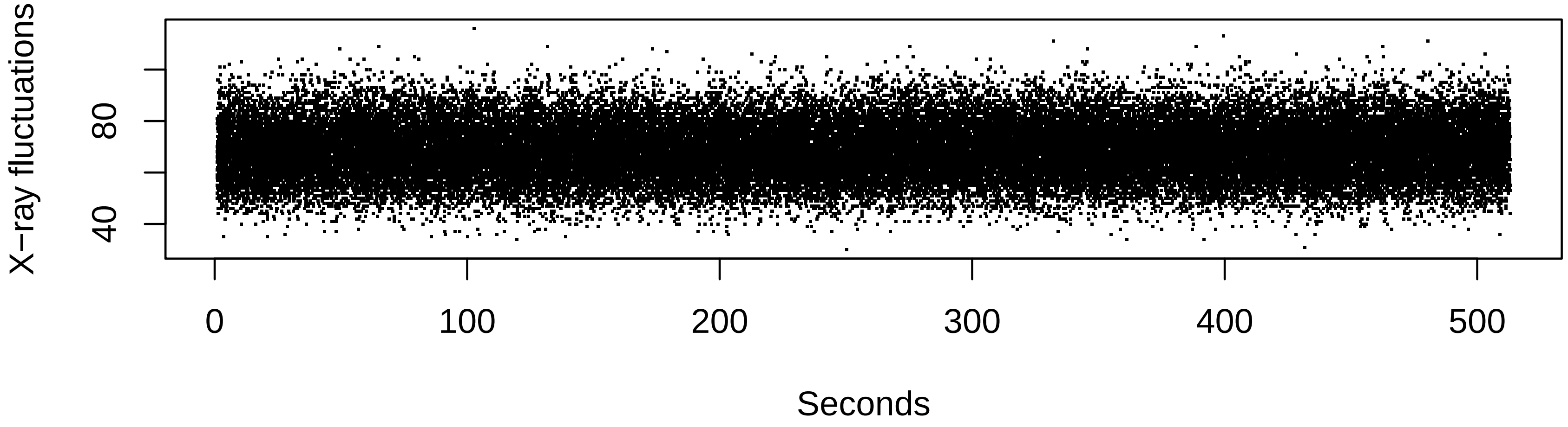
as $\{h_{j,l}\}$ is a **band-pass** filter over

$$f \in A_j = [-\frac{1}{2^j}, -\frac{1}{2^{j+1}}] \cup [\frac{1}{2^{j+1}}, \frac{1}{2^j}].$$

- $\nu_X^2(\tau_j)$ for **Haar** wavelets coincides with **Blackman–Tukey** pilot spectrum.
- Useful for **pure power law process** and processes with **time varying** spectra.

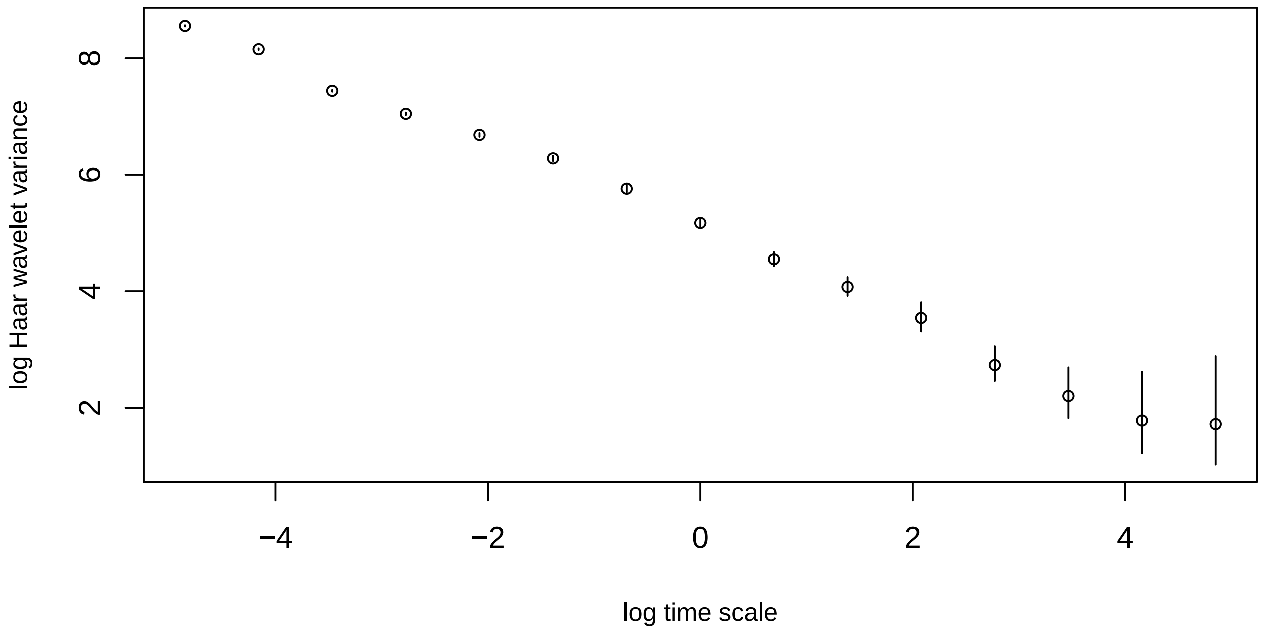
X-ray fluctuations for binary star

A 512 second fragment of the Ginga satellite data



Data records the X-ray counts in each of $65,526$ successive $1/128$ second intervals.

Wavelet variance analysis of X-ray fluctuations



The above plot shows **Haar wavelet variances** at log-log scales.

$\log(\nu_X^2(\tau_j))$ decays roughly **linearly** with $\log(\tau)$ over all scales.

This indicates a **power-law** model for X-ray fluctuations.

Slepian wavelet filters

For fixed j, let A_j be the **pass-band** of frequencies

$$A_j = [-\frac{1}{2^j}, -\frac{1}{2^{j+1}}] \cup [\frac{1}{2^{j+1}}, \frac{1}{2^j}]. \tag{3}$$

Let $\{\psi_m\}_{m=0}^{M-1}$ be a linear a band-pass filter with pass-band A_j .

$$\Psi(f) = \sum_{m=0}^{M-1} \psi_m e^{-i2\pi fm}.$$

we seek $\{\psi_m\}$ with the following properties:

$$\sum_m \psi_m = 0, \quad \sum_m \psi_m^2 = \frac{1}{2^j}, \quad \max_\psi \lambda(M,j) = \frac{\int_{A_j} |\Psi(f)|^2\, df}{\int_{-1/2}^{1/2} |\Psi(f)|^2\, df} = \frac{\psi^T Q_j \psi}{\psi^T \psi},$$

where the (s,t)th element of the $M \times M$ matrix Q_j is

$$Q_j(s,t) = \int_{A_j} e^{-i2\pi f(t-s)}\, df = \frac{\sin\left(\frac{2\pi(s-t)}{2^j}\right) - \sin\left(\frac{2\pi(s-t)}{2^{j+1}}\right)}{\pi(s-t)}.$$

Find maximum eigenvector $Q_j\psi = \lambda(M,j)\psi$ subject to $1^T\psi = 0.$ (4)

The continuous Slepian wavelet function

Set $M = c2^j$ for level j, where $2c$ is an integer **independent** of j. Then

$$Q_j(s,t) = \frac{\sin\left(\frac{2\pi c(s-t)}{M}\right) - \sin\left(\frac{\pi c(s-t)}{M}\right)}{\frac{M\pi(s-t)}{M}} = \frac{2}{M}\beta_c(f-f')$$

where $f = 2s/M - 1$ and $f' = 2t/M - 1$ so that $-1 \le f, f' < 1$ and

$$\beta_c(u) = \frac{\sin(c\pi u) - \sin(c\pi u/2)}{\pi u}.$$

Hence for large j, we have a **continuous** eigenvalue problem

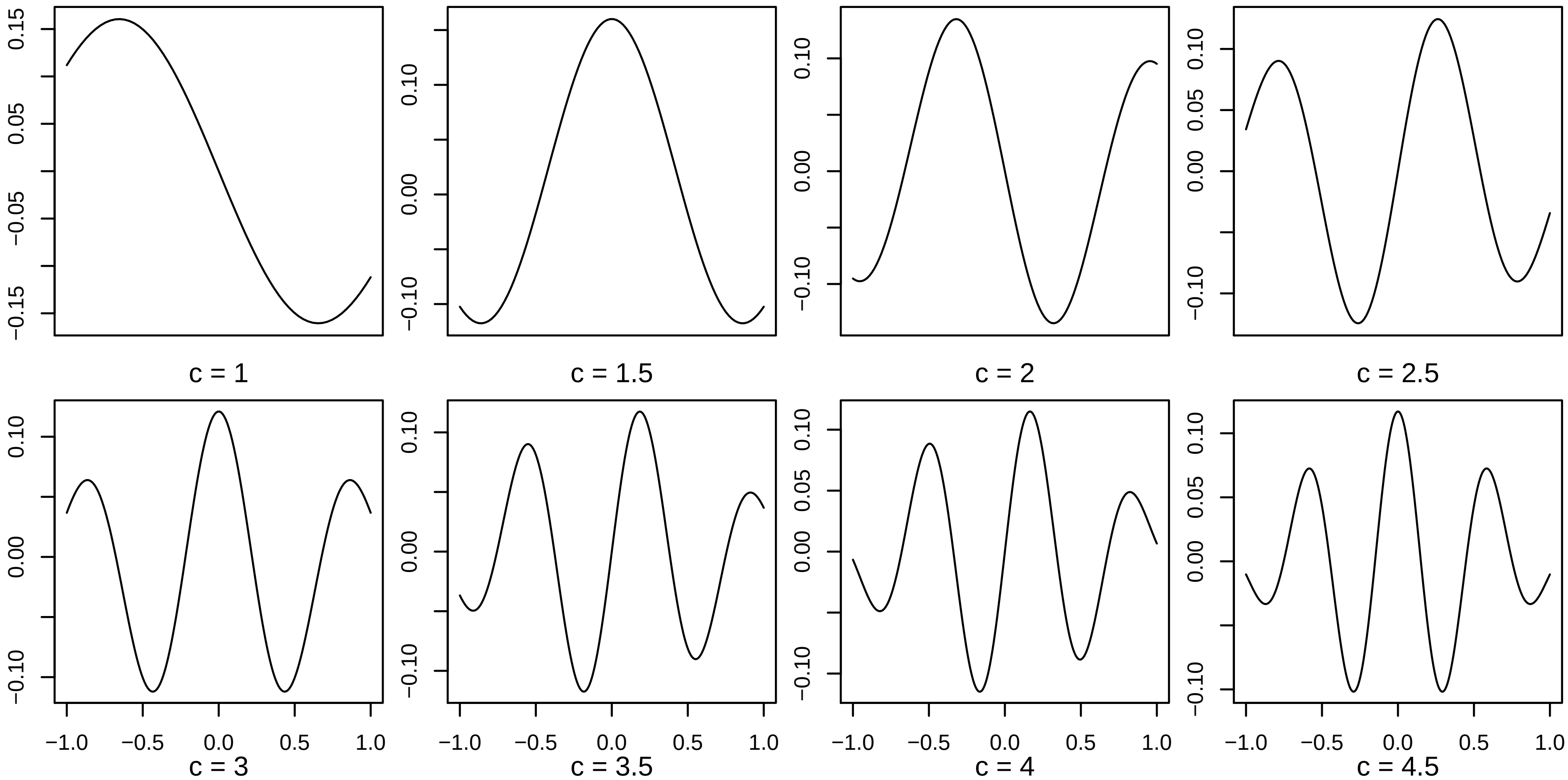
$$\int_{-1}^{1} \beta_c(f-f')\psi(f')\,df' = \lambda\psi(f) \tag{5}$$

subject to $\int \psi(f)\,df = 0$. Thus, at level j, we can approximate the discrete eigenvector

$$\frac{1}{\sqrt{M_j}}\psi(2s/M_j - 1), \quad s = \ 0,\ 1,\ \ldots\ M_j - 1. \tag{6}$$

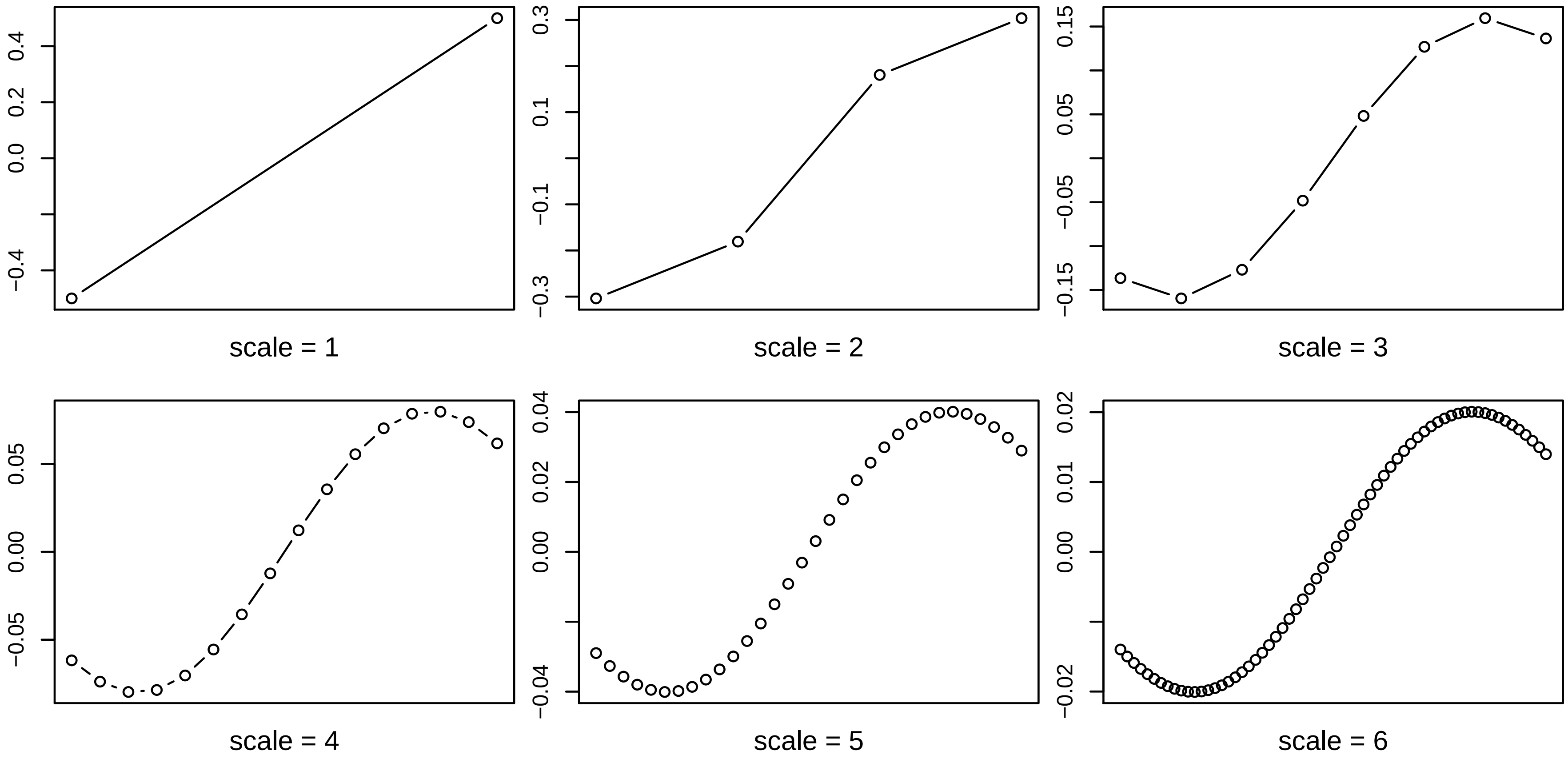
Important for **fast and efficient** computations!

The shape of the first eigenvector for different values of c



Obtained by computing **first eigenvector** with a large value of j.

Slepian wavelet filters at different scales for c=1



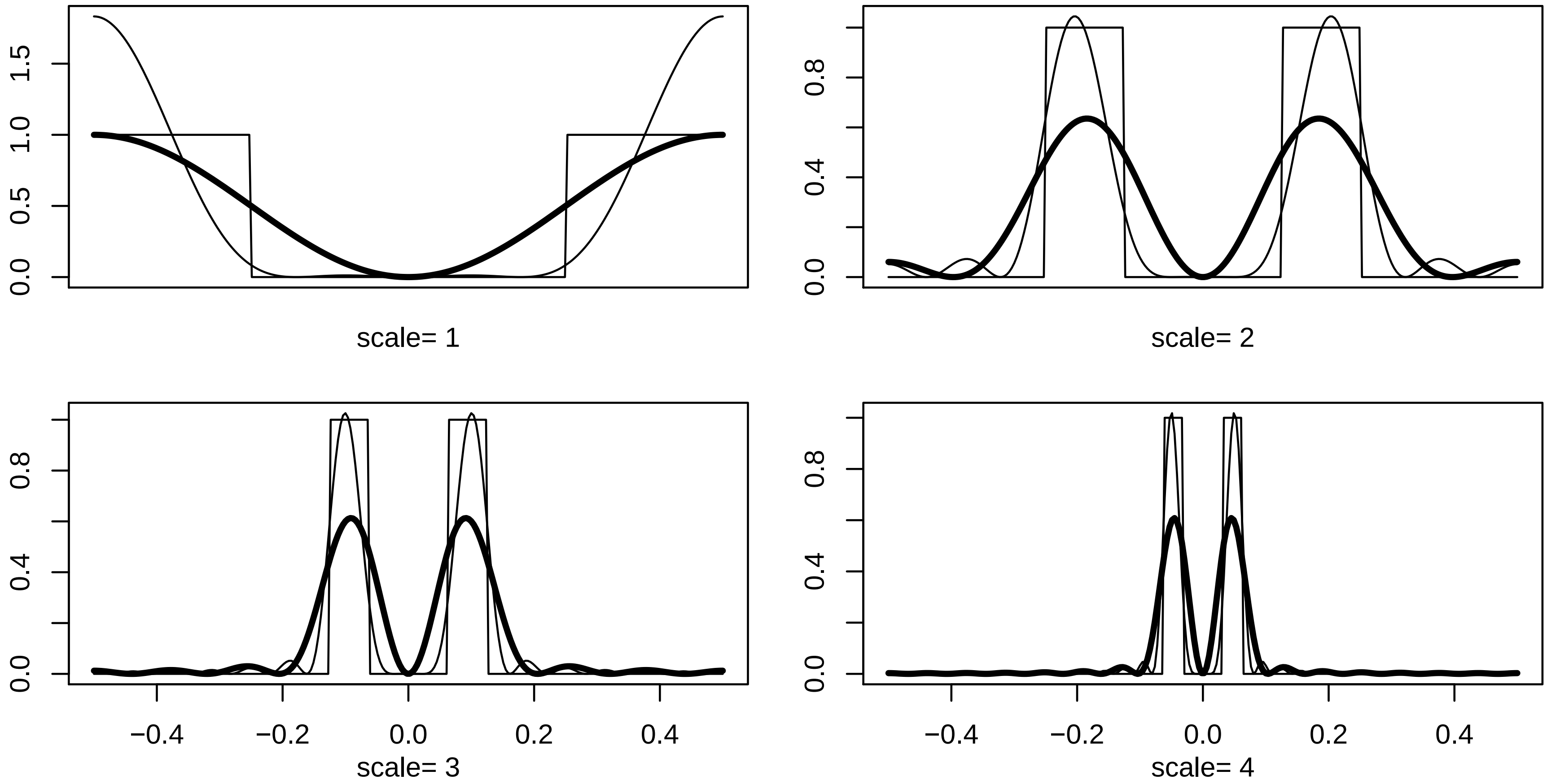
Approaches the continuos eigenvector fast.

Energy contained in the nominal pass-band

j	c=1	c=2	c=2.5	c=3		Haar	D(4)	D(6)	LA(8)
1	.818	.989	.998	1.00		.818	.871	.895	.909
2	.581	.787	.908	.951		.546	.641	.696	.733
3	.561	.781	.903	.947		498	.626	.691	.731
4	.557	.779	.902	.946		.487	.625	.691	.731
5	.556	.779	.902	.946		.487	.625	.691	.731
6	.556	.779	.902	.946		.487	.625	.691	.731
7	.555	.779	.902	.946		.487	.625	.691	.731
8	.555	.779	.902	.946		.487	.625	.691	.731

Compared to Daubechies wavelets, **energy leakage** is less for Slepian filters.

Squared gain functions of Slepian wavelets at different scales



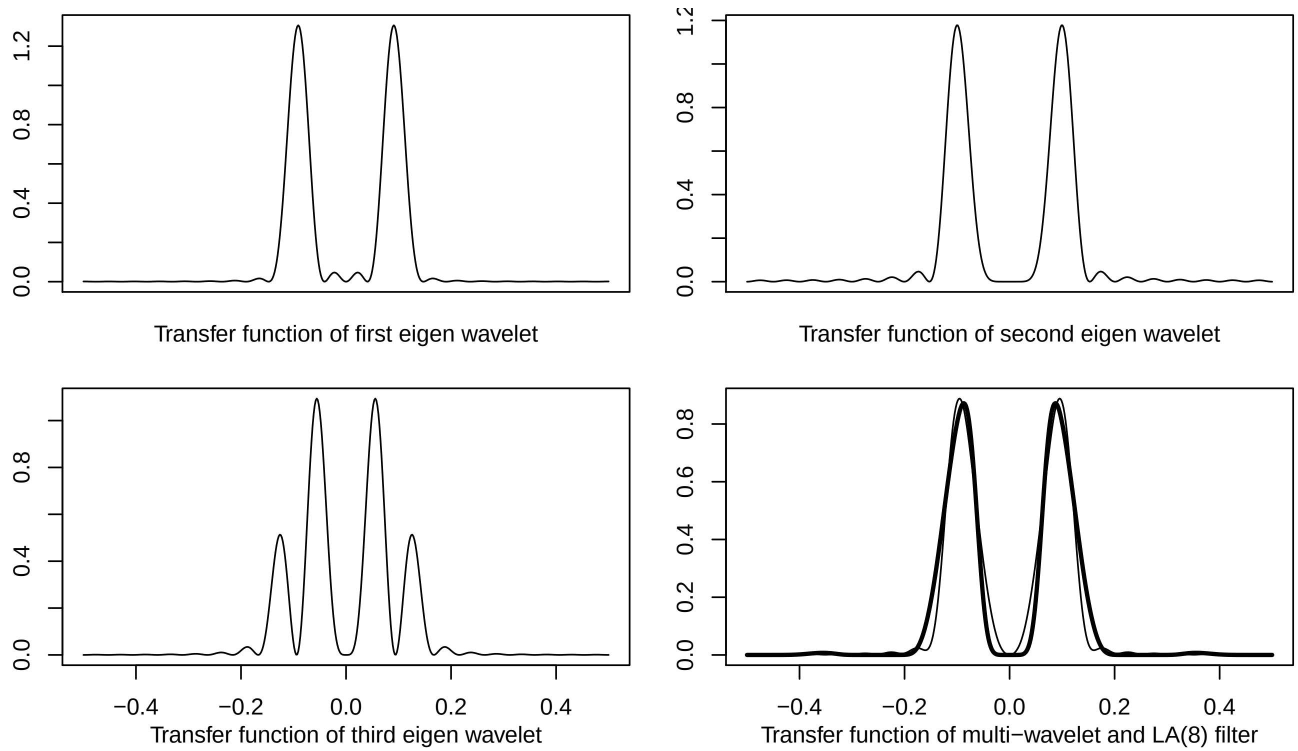
Thin line is for c=2, thick line is for c=1

Second eigenvalues for different scales and for various c

Level	c=1	c=2,	c=2.5	c=3
1	—	.712	.924	.974
2	.272	.751	.845	.910
3	.294	.754	.844	.913
4	.299	.755	.844	.914
5	.300	.755	.845	.914
6	.301	.755	.845	.914
7	.301	.755	.845	.914
8	.301	.755	.845	.914

Hence, we can possibly entertain more than one wavelet filters at each scale.

A multi-wavelet scheme for $c = 2.5$ with first three eigenvector



Plots of **squared gain functions** for the first three eigenvectors

Thin line in the last plot gives the **weighted transfer** function of these eigenvectors.

The thick line provides the squared gain function of LA(8) filter.

Weights are so chosen to **match the energy** of LA(8) filter.

Estimation of Slepian wavelet variance

Let $X_0, X_1, \ldots, X_{N-1}$ be the **observed** time series with spectral density $S(f)$.

The jth level **Slepian wavelet coefficient process** has the form

$$U_{j,t} \;=\; \sum_{u=0}^{M_j-1} \psi_{j,l} X_{t-l}.$$

The **Slepian wavelet variance** at dyadic scale $\tau_j = 2^{j-1}$ is given by

$$\mu_X^2(\tau_j) = \operatorname{var}\left(U_{j,t}\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\Psi_j(f)|^2 S(f)\, df$$

We can then obtain an **unbiased estimate** of $\mu_X^2(\tau_j)$ as:

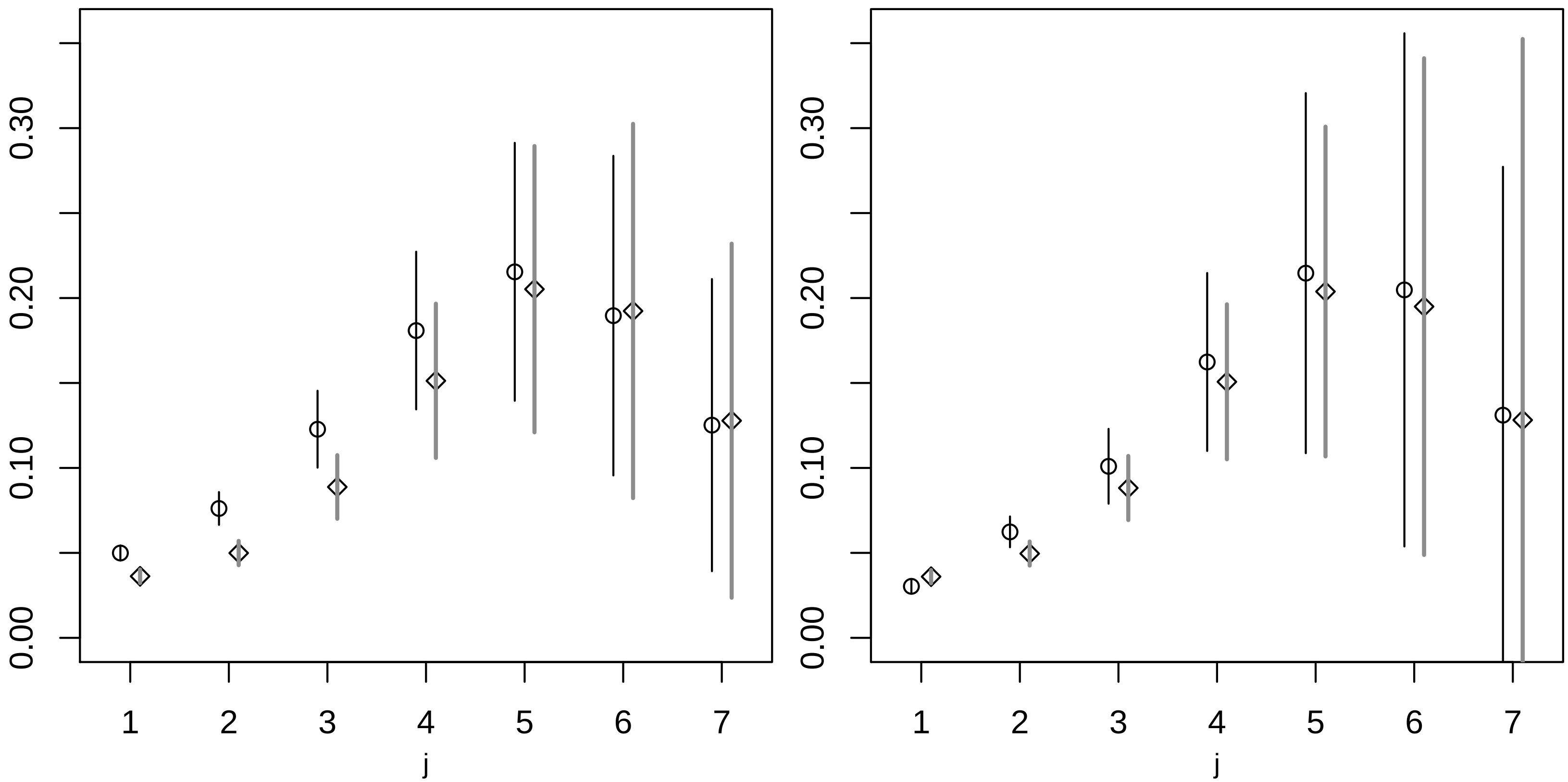
$$\hat{\mu}_X^2(\tau_j) = \frac{1}{N_j} \sum_{t=M_j-1}^{N-1} U_{j,t}^2, \qquad N_j \;=\; N - M_j + 1.$$

If $U_{j,t}$ is a mean zero stationary **Gaussian** series, and $\sigma_{U,j}^2 = \sum_t 2\operatorname{cov}^2(U_{j,0}, U_{j,t})$,

$$\sqrt{N_j}\Big(\hat{\mu}_X^2(\tau_j) - \mu_X^2(\tau_j)\Big) \longrightarrow_d \mathrm{N}\Big(0, \sigma_{U,j}^2\Big).$$

Simulation study for AR (1)

Plot of Daubechies (diamonds) and Slepian (circles) wavelet variances at various scales



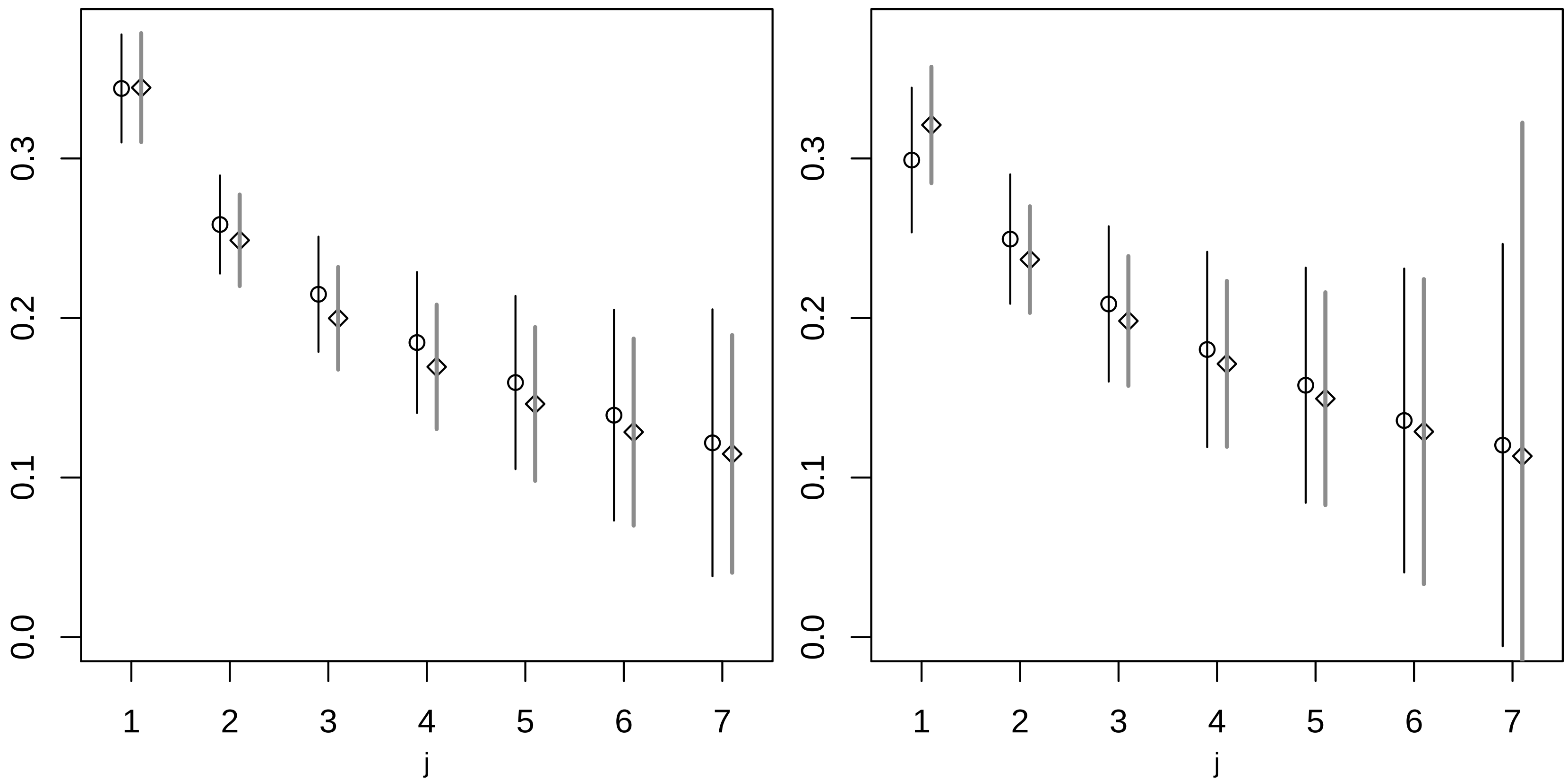
Sample size $N = 1024$. $\{X_t\}$ simulated from **AR(1)** with $s_{X,\tau} = \phi^{|\tau|}$, $\phi = 0.9$.

We generate 1000 **Monte Carlo realizations**.

Left plot uses $c = 1$ and Haar whereas **right plot** uses $c = 2.5$ and LA(8).

Simulation study for FD process

Plot of Daubechies (diamonds) and Slepian (circles) wavelet variances at various scales



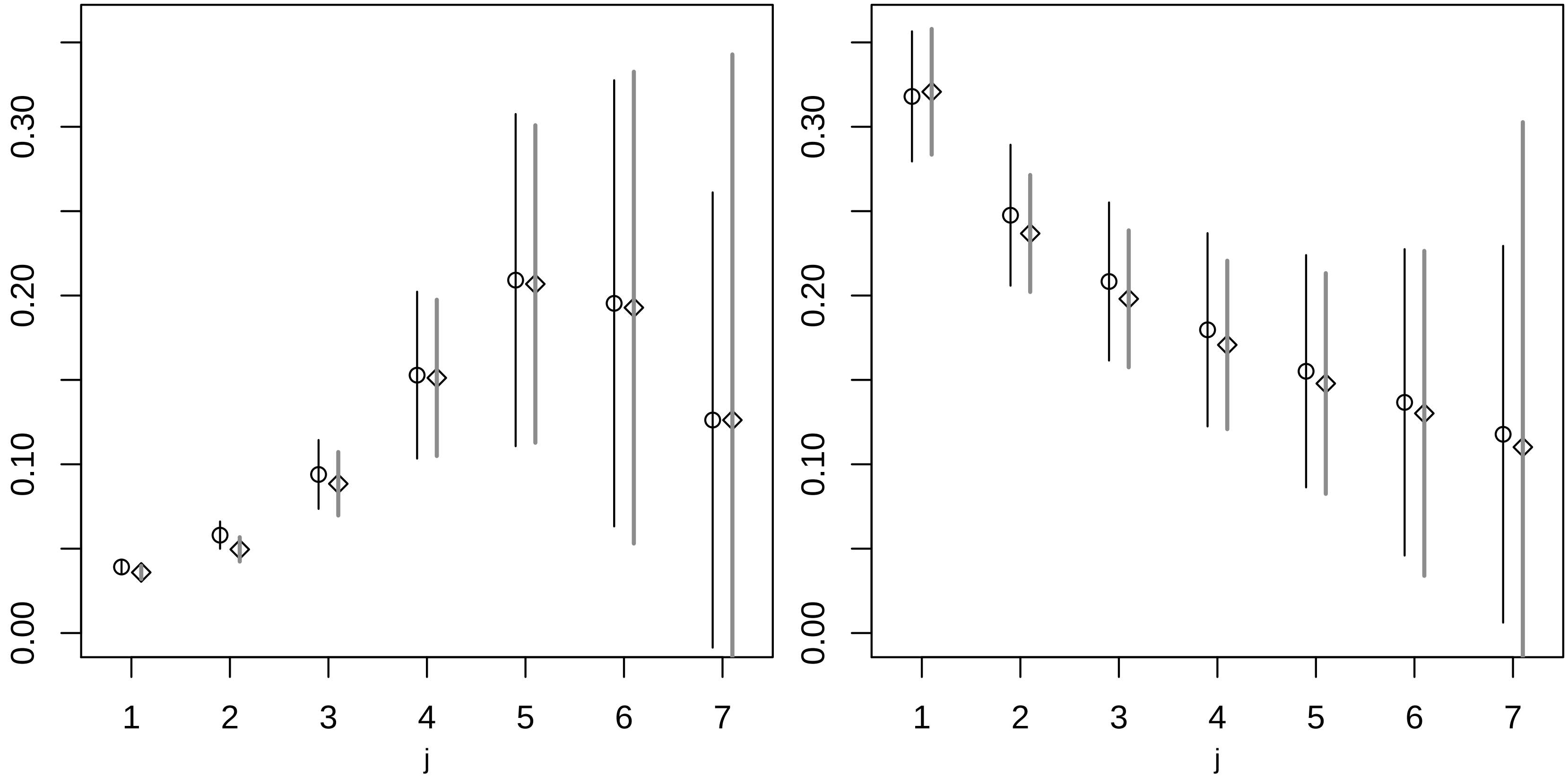
$N = 1024$, $\{X_t\}$ simulated from $\mathbf{FD}(0.4)$

We generate 1000 **Monte Carlo realizations**.

Left plot uses $c = 1$ and Haar whereas **right plot** uses $c = 2.5$ and LA(8).

Simulation study using a multi-wavelet scheme

Plot of Daubechies LA(8) (diamonds) and Slepian (circles) multi-wavelet variances

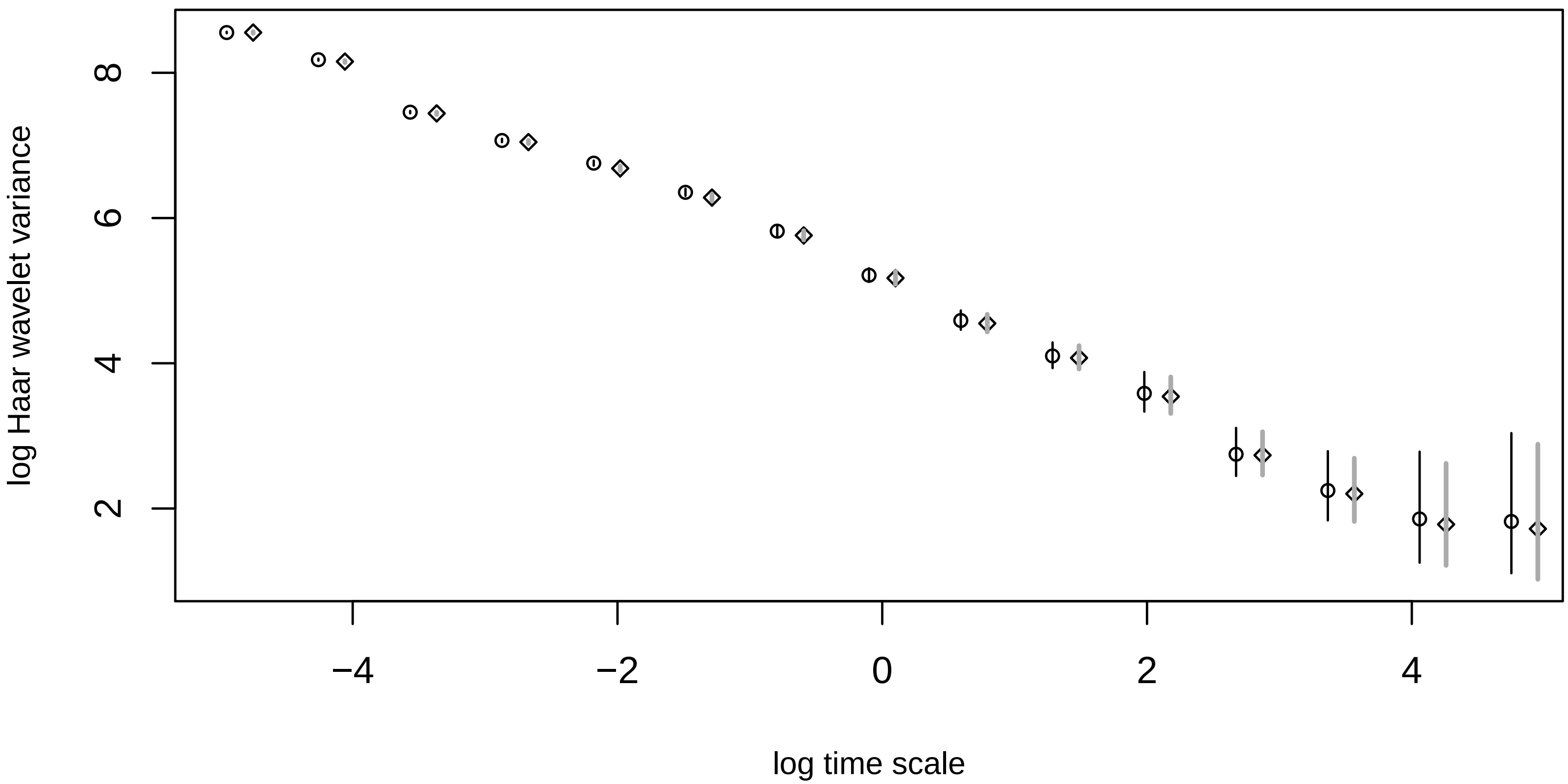


$N = 1024$, $\{X_t\}$ simulated either from AR(1) (left) or FD (right) process.

We generate 1000 **Monte Carlo realizations**.

We use $c = 2.5$ and first two eigenvectors with equal weights.

X-ray fluctuations for binary star



Circles and black lines gives **Slepian** wavelet variances.

Diamonds and gray lines indicate **Haar** wavelet variances.

Extension to irregularly sampled data

Observed data consist of **time series values** $X(t_0), X(t_1), \ldots X(t_{N+1})$.

Data **taken at** irregular time points $t_0, t_1, \ldots t_{N+1}$.

Many possibilities exist....

Can **Slepian filters adapt** to such **sampling**? What **time scales** to **consider**?

Sampling intervals are $\Delta_1 = t_1 - t_0, \Delta_2 = t_2 - t_1, \ldots, \Delta_n = t_n - t_{n-1}, \ldots$

And the **average sampling interval** is equal to

$$\bar{\Delta} = \frac{1}{N+1}(\Delta_1 + \Delta_2 + \ldots + \Delta_{N+1}) = \frac{t_{N+1} - t_0}{N+1}.$$

Hence it's natural to consider, for fixed j, the **pass-band** of frequencies

$$A_j = [-\frac{1}{2^j \bar{\Delta}}, -\frac{1}{2^{j+1}\bar{\Delta}}] \cup [\frac{1}{2^{j+1}\bar{\Delta}}, \frac{1}{2^j \bar{\Delta}}]. \tag{7}$$

$\Rightarrow$ **dyadic scales** $\tau_j = \bar{\Delta} 2^{j-1}$, $j = 1, 2, \ldots J$ with $N + 1 = 2^J + N_J$, $0 < N_J \leq 2^J - 1$.

Adaptive Slepian wavelet filters

For fixed j, A_j is the **pass-band** of frequencies

$$A_j = [-\frac{1}{2^j\bar{\Delta}}, -\frac{1}{2^{j+1}\bar{\Delta}}] \cup [\frac{1}{2^{j+1}\bar{\Delta}}, \frac{1}{2^j\bar{\Delta}}].$$

For each k, let $\{\psi_{k,m}\}_{m=0}^{M-1}$ be **adaptive to time points** $t_k, t_{k+1}, \ldots t_{k+M-1}$.

$$\Psi_k(f) = \sum_{m=0}^{M-1} \psi_{k,m} e^{-i2\pi f t_{k+m}}.$$

We **seek** $\{\psi_{k,m}\}$ with the **following properties**:

$$\sum_m \psi_{k,m} = 0, \quad \sum_m \psi_{k,m}^2 = \frac{1}{2^j\bar{\Delta}}, \quad \max_\psi \lambda(k,M,j) = \frac{\int_{A_j} |\Psi_k(f)|^2\, df}{\int_{-1/2}^{1/2} |\Psi_k(f)|^2\, df} = \frac{\psi_k^T Q_{k,j} \psi_k}{\psi_k^T \psi_k},$$

where the (s,t)th element of the $M \times M$ matrix $Q_{k,j}$ is

$$Q_{k,j}(m,m') = \int_{A_j} e^{-i2\pi f(t_{k+m}-t_{k+m'})}\, df = \frac{\sin\left(\frac{2\pi(t_{k+m}-t_{k+m'})}{2^j\bar{\Delta}}\right) - \sin\left(\frac{2\pi(t_{k+m}-t_{k+m'})}{2^{j+1}\bar{\Delta}}\right)}{\pi(t_{k+m}-t_{k+m'})}.$$

Find maximum eigenvector $Q_{k,j}\psi_k = \lambda(k,M,j)\psi_k$ subject to $1^T\psi_k = 0.$ (8)

Corresponding continuous eigenvalue problem

As before set $M = c2^j$ for level j, where $2c$ is an integer **independent** of j.

Assume that **sampling intervals** $\Delta_1, \ldots, \Delta_N$ arise form a **stationary** process,

for which **strong invariance principle** holds (e.g, **Renewal sampling** scheme). Then

$$\frac{t_{k+m} - t_{k+m'}}{M} \stackrel{d}{=} \frac{t_m - t_{m'}}{M} = \frac{t_m}{m}\frac{m}{M} - \frac{t_{m'}}{m'}\frac{m'}{M} \rightarrow_d \tfrac{1}{2}\mu(f - f').$$

where $f = 2m/M - 1$ and $f' = 2m'/M - 1$ so that $-1 \le f, f' < 1$ and $\mu = \mathrm{E}\,\bar{\Delta}$.

And

$$MQ_{k,j}(m, m') = M\frac{\sin\!\left(\frac{2\pi(t_{k+m} - t_{k+m'})}{2^j\bar{\Delta}}\right) - \sin\!\left(\frac{2\pi(t_{k+m} - t_{k+m'})}{2^{j+1}\bar{\Delta}}\right)}{\pi(t_{k+m} - t_{k+m'})} \rightarrow_d 2\mu\beta_c(f - f')$$

$\Rightarrow$, for large j, we have the **same continuous eigenvalue problem as before**. i.e.,

$$\int_{-1}^{1} \mu\beta_c(f - f')\psi_k(f')\,df' = \lambda\psi_k(f),$$

Thus, at level j, we can approximate the discrete eigenvector

$$\frac{1}{\sqrt{M_j}}\psi\Big(2\frac{t_{m+k} - t_k}{t_{M+k} - t_k} - 1\Big), \quad m = 0, 1, \ldots M - 1, \quad k = 0, 1, \ldots. \tag{9}$$

Estimation of Slepian wavelet variance for irregular data

Let $\{\psi_{j,k,m}\}$ be **adaptive wavelet filters** for time points $t_k, t_{k+1}, \ldots t_{k+M_j-1}$.

Define Slepian **wavelet coefficients** index by scale τ_j and shift k as follows:

$$U_{j,k} = \sum_{u=0}^{M_j-1} \psi_{j,k,u} X(t_{k+u}), \quad k = 0, 1, \ldots N - M_j + 2.$$

A **naive estimate** of overall energy (wavelet variance) at scale τ_j is then given by

$$\hat{v}^2(\tau_j) = \frac{1}{N - M_j + 2} \sum_{k=0}^{N-M_j+2} U_{j,k}^2$$

Conditional on the time points $t_k, t_{k+1}, \ldots t_{k+M_j-1}$,

$$\mathrm{E}\,(U_{j,k}^2) = \int_{-\infty}^{\infty} \left|\Psi_{j,k}(f)\right|^2 S_X(f)\, df, \quad \Psi_{j,k}(f) = \sum_{m=0}^{M_j-1} \psi_{j,k,m} e^{i2\pi f t_{m+k}}.$$

If spectral density is **approximately** constant within A_j, that is, $S(f) = \nu^2(\tau_j) 2^j \bar{\Delta}$,

$$\mathrm{E}\,(U_{j,k}^2) \approx \int_{A_j} \left|\Psi_{j,k}(f)\right|^2 \nu^2(\tau_j) 2^j \bar{\Delta}\, df = \lambda(k, M_j, j)\nu^2(\tau_j) \approx \nu^2(\tau_j). \qquad (10)$$

Large sample properties of $\hat{v}^2(\tau_j)$

$U_{j,k}$ form a **mean zero stationary** time series if

(i) $X(t_0), \ldots, X(t_{N+1})$ is a realization of a stationary stochastic process,

and (ii) the sampling times obey a stationary point process

Hence, under a**ppropriate regularity conditions**,

$$\sqrt{N - M_j + 1}\Big(\hat{v}^2(\tau_j) - \mathrm{E}\,\hat{v}^2(\tau_j)\Big) \longrightarrow_d \mathrm{N}\Big(0, S^2_{U^2_{j,k}}(0),\Big).$$

where $S^2_{U^2_{j,k}}(0)$ is spectral density of $U^2_{j,k}$ at 0.

How to **estimate** this **spectrum at origin**. Use **multitaper spectral method**

Let $\{\lambda_{k,l}\}_{k=0}^{N-M_j-2}$ for $l = 0, 1, \ldots, L-1$ be first L orthonormal **Slepian tapers**.

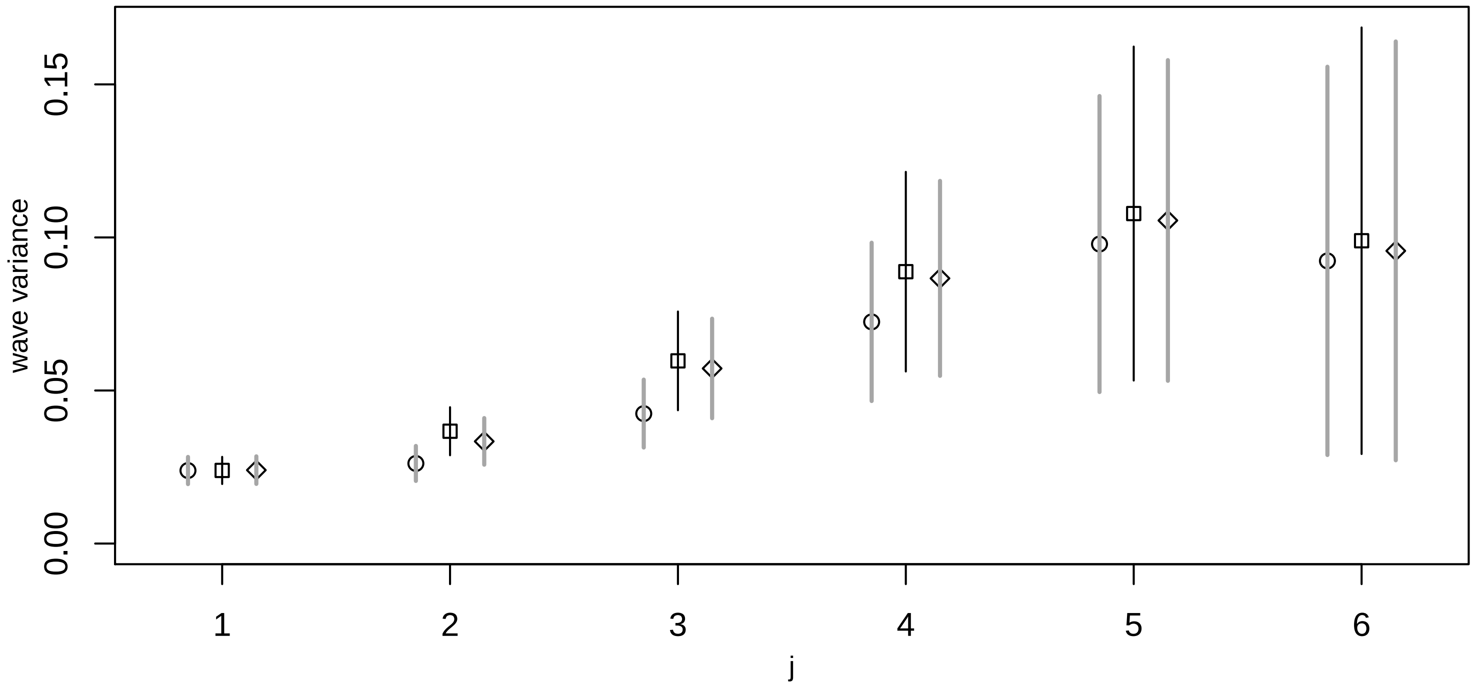
$$J_{j,l} = \sum_{k=0}^{N-M_j-1} \lambda_{k,l} U^2_{j,k}, \qquad \lambda_{+,l} = \sum_{k=0}^{M_j-1} \lambda_{k,l}.$$

Estimate $S^2_{U^2_{j,k}}(0)$ by

$$\hat{\sigma}^2_j = L^{-1} \sum_{l=0}^{L-1} (J_{j,l} - \widetilde{u}_j \lambda_{+,l})^2, \qquad \widetilde{u}_j = \sum_{l=0,2,\ldots}^{L-1} J_{j,l}\lambda_{+,l} \Big/ \sum_{l=0,2,\ldots}^{L-1} \lambda^2_{+,l}.$$

Simulation study for Ornstein–Uhlenbeck process

Slepian wavelet variances at various scales for three different sampling schemes



$N = 512$ and $\mu = 2$. Data simulated from Ornstein–Uhlenbeck process with

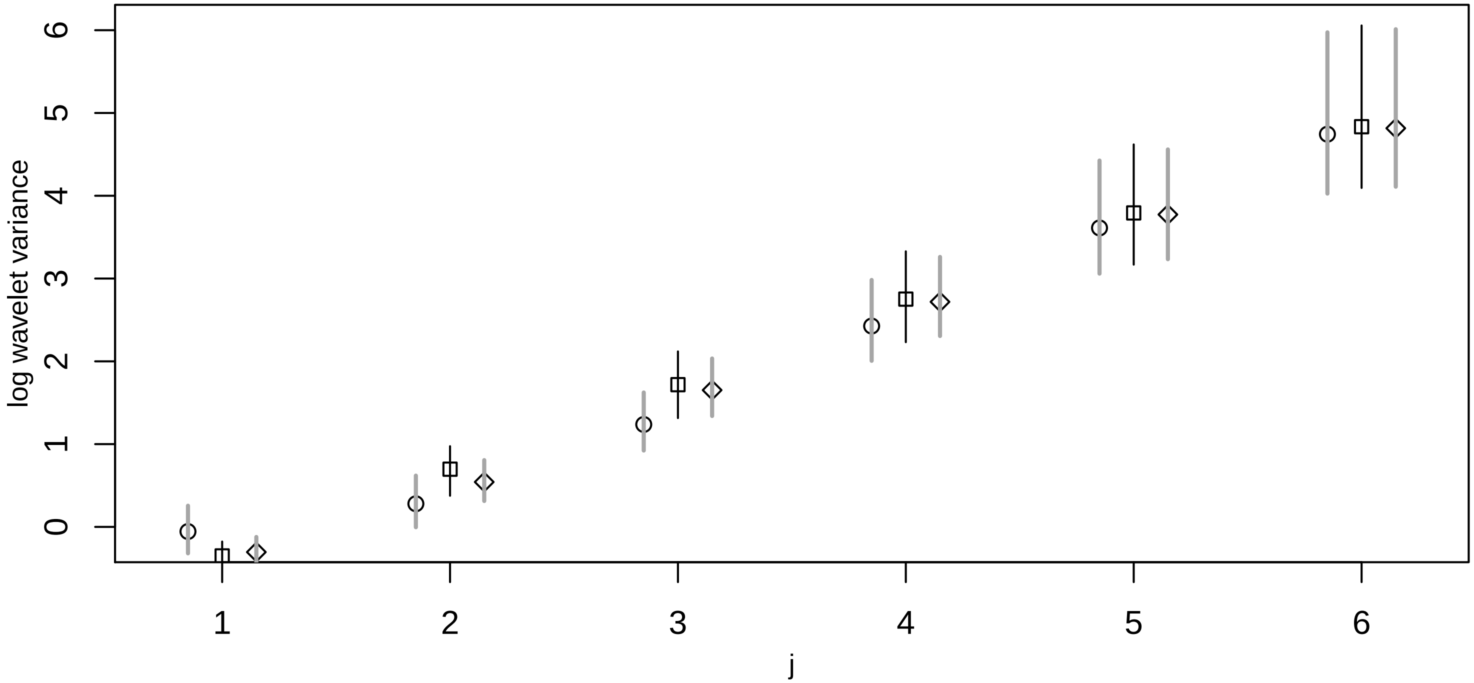
$$\text{cov}(X(t), X(s)) = e^{-\rho|t-s|}, \quad \rho = -\log(0.9)/2.$$

We generate 1000 **Monte Carlo realizations**.

Circles for **Poisson process** sampling, **boxes** for even integer **lattice**, and **diamonds** for a **renewal process** sampling.

Simulation study for fractional Brownian motion

Slepian wavelet variances at various scales for three different sampling schemes



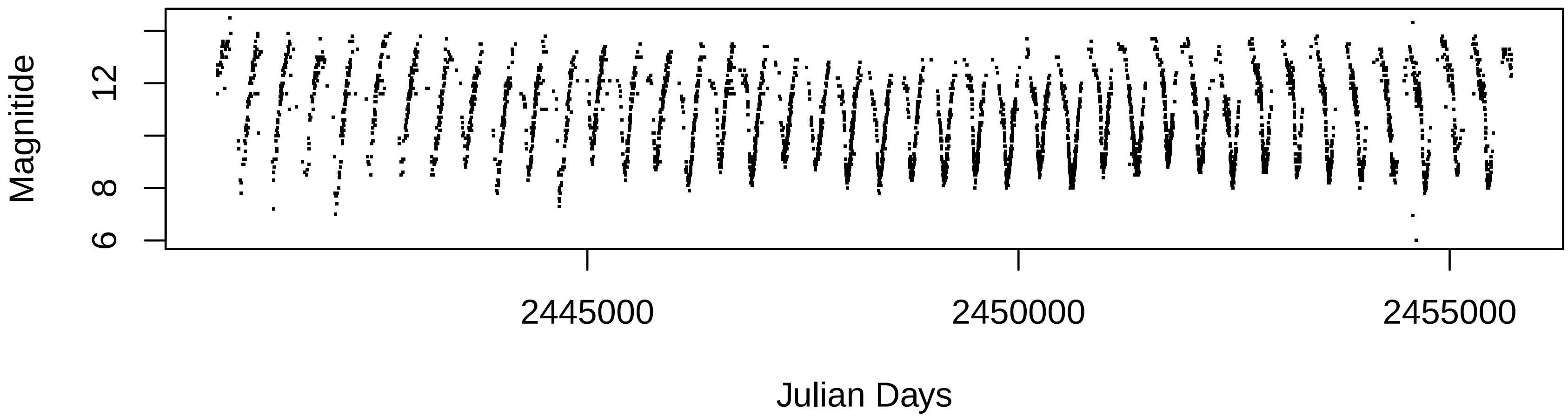
$N = 512$ and $\mu = 2$. Data simulated from fractional Brownian motion with

$$\mathrm{cov}(X(t), X(s)) = |t|^{2H} + |s|^{2H} - |t-s|^{2H}, \quad H = .75.$$

We generate 1000 **Monte Carlo realizations**.

Circles for **Poisson process** sampling, **boxes** for even integer **lattice**, and **diamonds** for a **renewal process** sampling.

Light curve data of S Ser

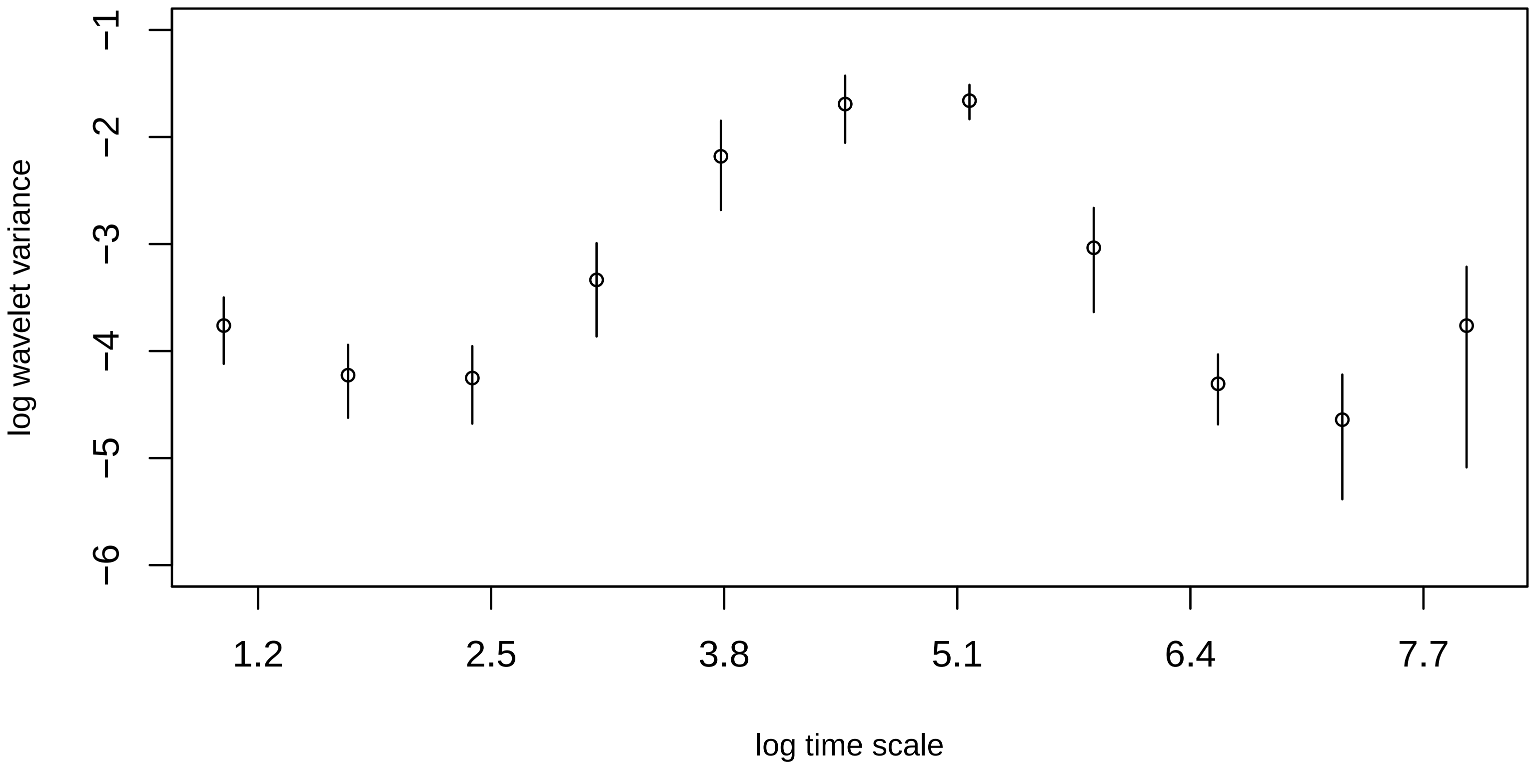


S Ser is a **Mira variable star** in the constellations Serpentis

Data taken directly from website of **AAVSO**.

Magnitude ranges from **7** to 14:1V with a **pulsating period** of about 371.84 days.

Wavelet variance plot for light curve data of S Ser



- **Irregularly** sampled data with **average sampling interval** of 2.743 days.
- **Peak** at time scale $j = 2^6 \times 2.743 = 175.54$ days **suggests a nominal period** of 351.08 days.
- Significant variance in the light curve over scales eight years and longer.

Some new ideas on estimation of wavelet spectra

$\psi(u), -\infty < u < \infty$ a **wavelet function** (e.g., Daubechies, or Slepian wavelet).

$\psi(u)$ **compactly supported** on $[0, \infty)$, integrates to 0, $\psi^2(u)$ integrates to 1, and

$$\int_0^\infty |\Psi(f)|^2 f^{-1}\, df < \infty, \quad \text{where} \quad \Psi(f) = \int_{-\infty}^\infty \psi(u) e^{-2i\pi f u}\, du.$$

The **continuos time** wavelet transform process is

$$W(\lambda, t) = \int_{-\infty}^\infty \psi_\lambda(u - t) X(u)\, du, \quad \text{where} \quad \psi_\lambda(u) = \lambda^{-\frac{1}{2}} \psi(u/\lambda), \quad \text{and} \quad \lambda > 0, -\infty < t < \infty$$

We can now define **wavelet spectra** as variance of this time series

$$\nu^2(\lambda) = \text{var}\Big(W(\lambda, t)\Big).$$

Increments of $X(t)$ is stationary with **variogram** function $\gamma(t) = \frac{1}{2} E\{X(t) - X(0)\}^2$

$$\Rightarrow \qquad \nu^2(\lambda) = -\int_{-\infty}^\infty \int_{-\infty}^\infty \psi_\lambda(u) \lambda(u') \gamma(u - u')\, du du'.$$

Goal is to estimate $\nu^2(\lambda)$ for **irregularly** sampled data.

Some new ideas on estimation of wavelet spectra

We define a **stationary stochastic** process V_k, $k = 1, 2, \ldots$, by

$$V_k = -\tfrac{1}{2}\sum_{l=1}^{M}\sum_{l'=1}^{M}\psi_\lambda(t_{l+k}-t_k)\psi(t_{l'+k}-t_k)\omega_{l,l'}(t_{l+k}-t_k, t_{l'+k}-t_k)\Big(X(t_{l+k})-X(t_{l'+k})\Big)^2,$$

where the **weight functions** are so restricted that $\omega_{l,l'}(u,u') = \omega_{l',l}(u,u')$.

Can we find weights $\omega_{l,l'}$ so that $\mathrm{E}\, V_k = \nu^2(\lambda)$ **???**

YES, if we **choose**, for $l < l'$,

$$\omega_{l,l'}^{-1}(u,u') = \begin{cases} \frac{1}{2}M(M-1)c_{l,l'}(u,u') & \text{if } u < u', \\ \frac{1}{2}M(M-1)c_{l',l}(u,u') & \text{if } u > u' \end{cases} \qquad (11)$$

where

$$c_{l,l'}(u,u') = p_{l,l'}(u, u'-u), \quad u < u',$$

where $p_{l,l'}(u, u'-u)$ is the **bivariate pdf** of $t_l - t_0$ and $t_{l'} - t_l$.

Further simplifications occurs for renewal for Poisson process samplings.

$\Rightarrow$ Estimation of wavelet spectra requires bivariate density estimations!!

Estimation and inference of wavelet spectra

Estimate **bivariate pdfs** of $t_l - t_0$ and $t_{l'} - t_l$, and weights $\hat{\omega}_{l,l'}(t_{l'+k} - t_k, t_{l+k} - t_k)$.

Define

$$\hat{V}_k = -\tfrac{1}{2} \sum_{l=1}^{M} \sum_{l'=1}^{M} \psi_\lambda(t_{l+k} - t_k)\psi(t_{l'+k} - t_k)\hat{\omega}_{l,l'}(t_{l+k} - t_k, t_{l'+k} - t_k)\Big(X(t_{l+k}) - X(t_{l'+k})\Big)^2,$$

Estimate $\nu^2(\lambda)$ by

$$\hat{\nu}^2(\lambda) = \frac{1}{N - M - 1} \sum_{k=1}^{N-M+1} \hat{V}_k$$

$\hat{\nu}^2(\lambda)$ will be an **almost unbiased** estimator of $\nu^2(\lambda)$.

Under regularity conditions,

$$\sqrt{N - M + 1}\Big(\hat{\nu}^2(\lambda) - \nu^2(\lambda)\Big) \longrightarrow_d \mathrm{N}\Big(0, S_{V_k}(0)\Big).$$

Use **Multi-taper estimator** to estimate spectral density of V_k at 0 that is $S_{V_k}(0)$.

Choices of scales and M: consider dyadic scales $2^{j-1}\mu$, and $M_j = c2^j$ as before.

Extends the work of Mondal and Percival (2008) on **missing observations**.

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