The probability density function of constrained correlation functions

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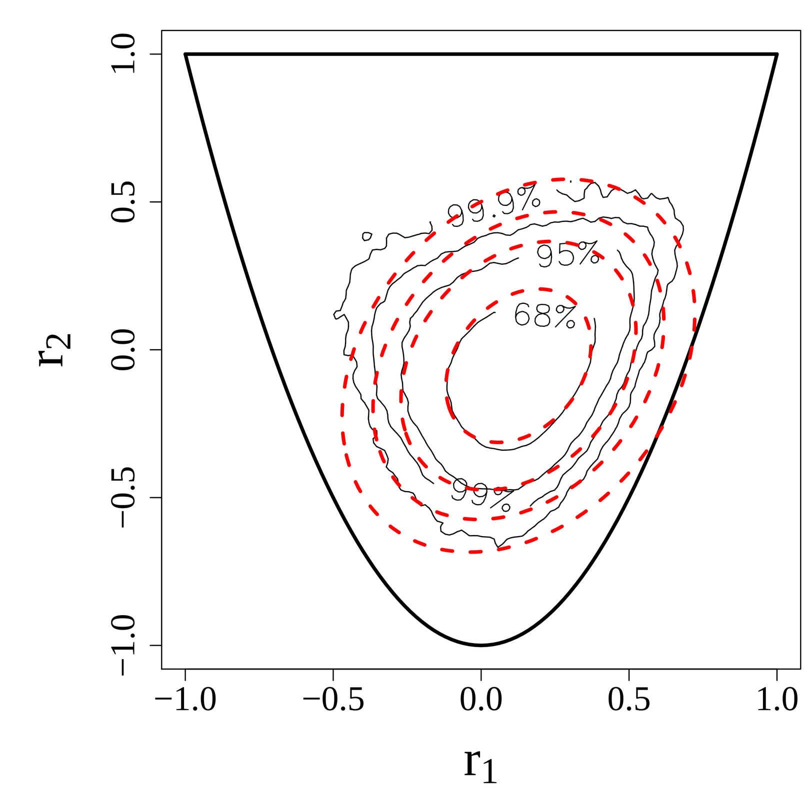
Abstract

The likelihood function of correlation functions needs to be known whenever they are used for inference about cosmological parameters. It is usually approximated as a multivariate Gaussian, which is not necessarily a good approximation, as can be seen from the existence of constraints on correlation functions (see [1]) – thus, a better approximation for the likelihood of correlation functions is required.
For a 1-D Gaussian field, we derived analytically the univariate and bivariate likelihood which can deviate very strongly from Gaussians. Furthermore, based on the constraints and the exact univariate likelihood, we constructed a quasi-Gaussian ansatz for the multi-variate correlation likelihood which (1) strictly obeys the constraints, (2) yields an approximate Gaussian in cases where the Gaussian approximation for the likelihood holds, and, if this is not the case, (3) provides a much better approximation than the Gaussian, as demonstrated with simulations; finally, (4) it provides a significantly better description than the straightforward copula approach. We apply this new correlation likelihood to a toy model to demonstrate its performance.

Constraints on correlation functions

As shown in [1], correlation functions $\xi(x)$ of a random field cannot take arbitrary values, but are subject to constraints, originating from the non-negativity of the power spectrum $P(k)$. The constraints can be written in terms of the correlation coefficients $r_n \equiv \xi(nx)/\xi(0)$ as $r_{n\mathrm{l}} \leq r_n \leq r_{n\mathrm{u}}$, where the upper and lower boundaries are functions of the r_i with $i < n$.
The figure shows how the constraints (in this case: $-1 + 2r_1^2 \leq r_2 \leq 1$, shown as black, solid lines) are obeyed by the true $p(r_1, r_2)$ (grey contours), and how the best-fitting Gaussian (red, dashed contours) spills into the forbidden region.

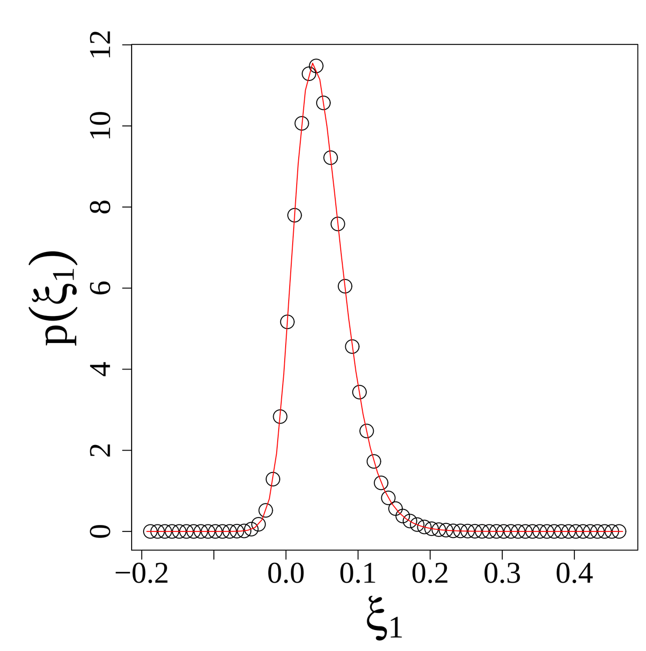


Analytical calculation of $p(\xi)$

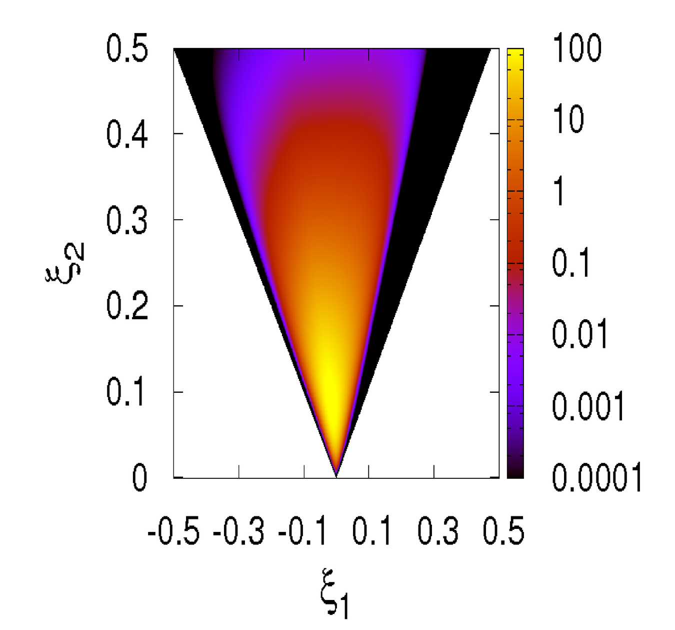
For a one-dimensional, finite, real Gaussian random field, we have analytically derived the uni- and bivariate probability distribution functions of its correlation functions. Starting from a Fourier expansion, we obtain the characteristic function. From this, integration with the theorem of residues yields the likelihood function as

$$p(\xi) = \sum_{n=1}^{\infty} \mathcal{H}_n \mathrm{e}^{-\xi/(2C_n)} \frac{1}{2C_n} \prod_{m \neq n} \frac{1}{1 - \frac{C_m}{C_n}}$$

where the $C_n = \sigma_n^2 \cos(k_n x)$ come from the power spectrum (discrete modes with dispersion σ_n) and $\mathcal{H}_n = H(\xi)H(C_n) - H(-\xi)H(-C_n)$ is a Heaviside function term implementing the constraints. The bivariate distribution is similar, but more complicated.
The detailed derivation and several consistency checks can be found in [2]. We also found the analytical distributions to agree well with simulations.

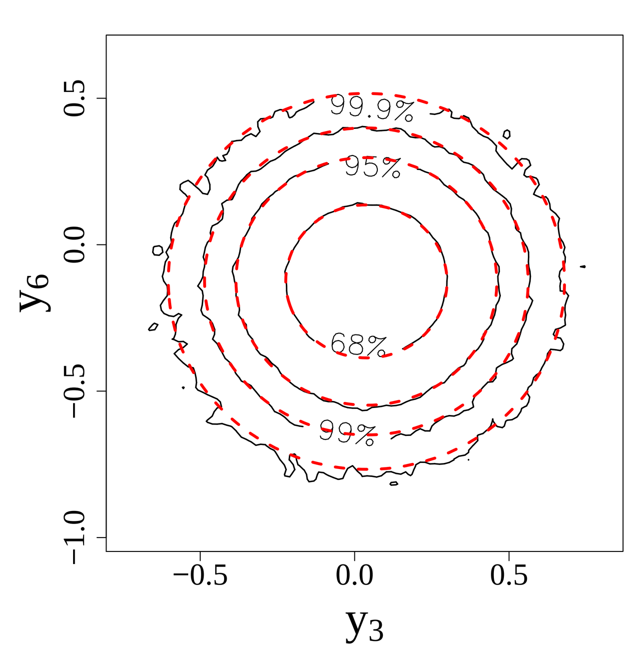


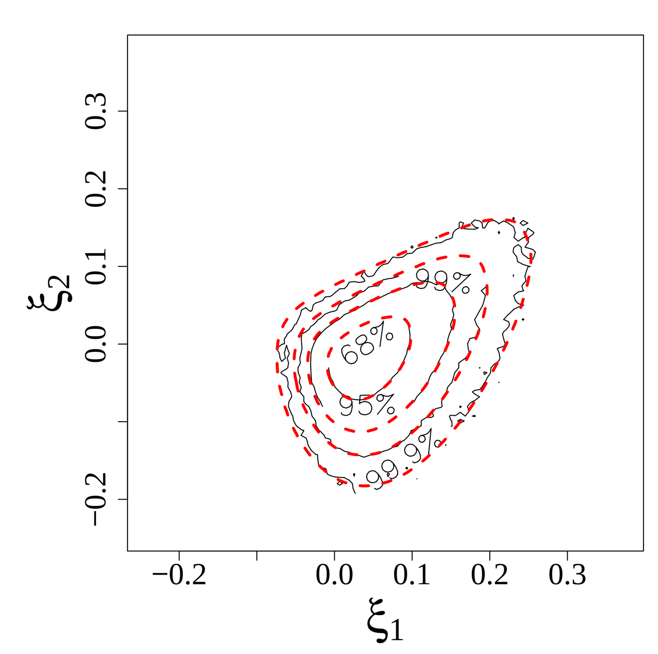
Analytical univariate distribution (line) compared to simulations (points)



Bivariate distribution $p(\xi_1(x), \xi_2(0))$ (color scale) obeying the constraints

The quasi-Gaussian approach





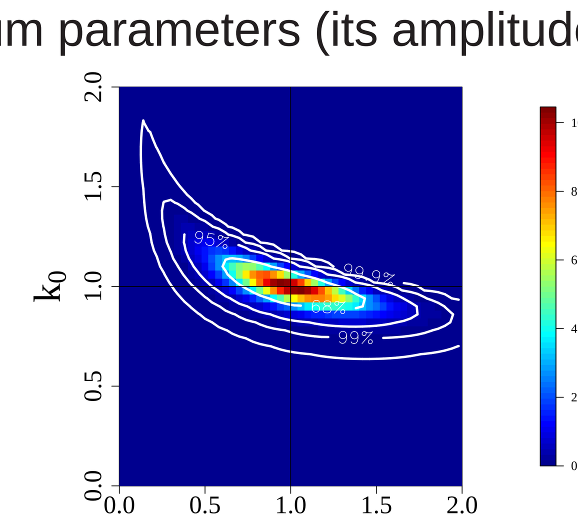
For higher multivariates, we can still efficiently construct a new, "quasi-Gaussian" likelihood that also obeys the constraints. For that purpose, we transform the correlation function to a new unbounded quantity

$$y_n = \operatorname{atanh} \frac{2r_n - r_{n\mathrm{u}} - r_{n\mathrm{l}}}{r_{n\mathrm{u}} - r_{n\mathrm{l}}}.$$

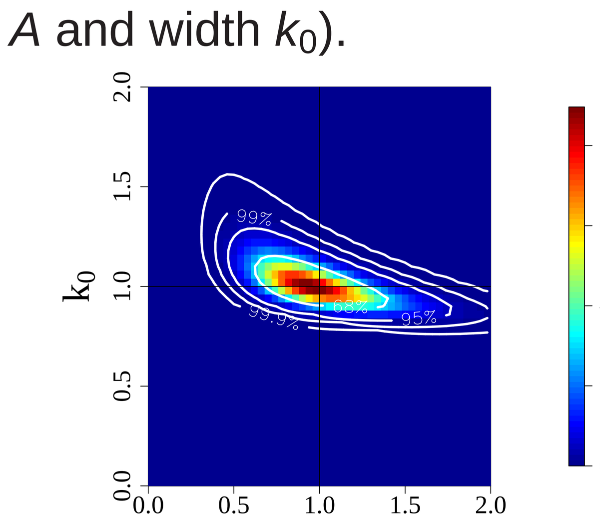
Using a Gaussian likelihood for y is by far better justified than for ξ, as illustrated by the left-hand figure (the black contours come from simulations, the best-fitting Gaussian is shown in red). Transforming the Gaussian back to ξ-space gives a good approximation for the likelihood (right-hand figure, see [3] for details and more quantitative results).
The mean and covariance matrix of the Gaussian in generally depend on ξ_0, so far we obtain this dependence phenomenologically via simulations – however, we also found formulae for calculating it analytically.

Impact on parameter estimation

To test how the new likelihood performs in a Bayesian analysis compared to the Gaussian likelihood, we construct a toy model: From simulated realizations of the correlation function on a field with a Gaussian power spectrum, we try to gain inference on the power spectrum parameters (its amplitude A and width k_0).
Already for this simple toy model, the impact of the more accurate likelihood on the posterior is visible. The difference may be larger for a different choice of power spectrum.



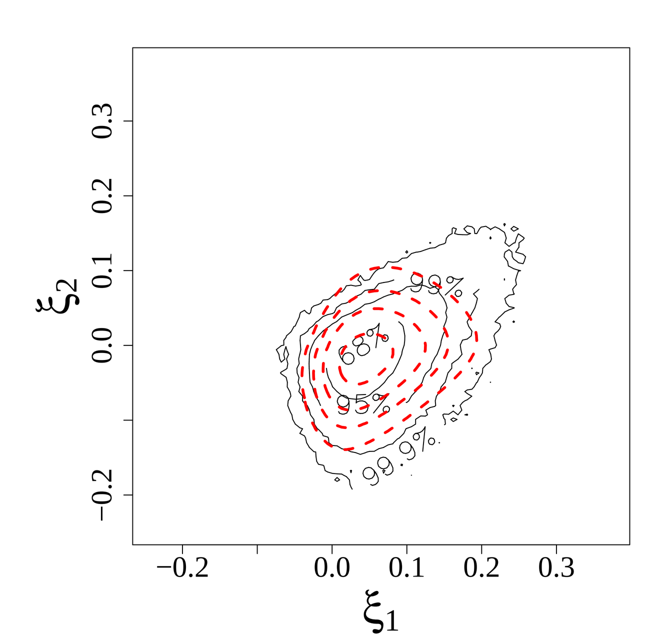
Posterior for the power spectrum parameters in the case of the Gaussian likelihood...



...and the quasi-Gaussian likelihood.

A copula approach

As an alternative way to construct likelihood functions, [4] – [6] introduce to cosmology the copula, which couples univariate distributions to get a multivariate PDF. Since in our case, the exact univariate PDF of ξ is known thanks to [2], using a copula approach seems to be an obvious step.
Coupling the analytical univariate $p(\xi)$ with a Gaussian copula yields a multi-variate likelihood that is not in good agreement with simulations, as the figure shows. Thus our quasi-Gaussian approximation should be favored also over the copula likelihood – of course, the accuracy of the latter might improve if a more realistic coupling was found.



$p(\xi_1, \xi_2)$ from the simulations (black contours) and the Copula likelihood (red, dashed contours)

References

[1] P. Schneider & J. Hartlap, A&A 504:705–717 (2009)
[2] D. Keitel & P. Schneider, arXiv:1105.3672, submitted to A&A (2011)
[3] P. Wilking & P. Schneider, in preparation
[4] R. Scherrer et al., ApJ, 708, L9 (2010)
[5] M. Sato et al., Phys. Rev. Lett., 105, 251301 (2010)
[6] M. Sato et al., Phys. Rev. D, 83, 023501 (2011)